"El saber de mis hijos
hará mi grandeza"

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Geometry of Lagrangian Mechanics

T E S I S

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Contents

Introduction

Lagrangian mechanics appeared in 1788 in the treatise *Mécanique Analytique* by Joseph-Louis Lagrange as an alternative to the newtonian formulation of mechanics. Lagrange's work attempts to detach the problem solving from the use of diagrams and intuition, inherent in Newton's work. Therefore, lagrangian formulation is a rather analytical approach to mechanics, i.e., using $\mathbb{R}-$valued functions and differential equations. The laws of movement in the new context become the Euler-Lagrange (EL) equations. One of the purposes in this thesis is to translate Lagrange's formalism to geometric terms, i.e., to the language of vector fields and differential forms on manifolds instead of differential equations, i.e., an *intrinsic* formulation of the EL equation.

A situation where lagrangian mechanics manifests its advantages over the newtonian, is in the presence of *holonomic constraints*. In this case, if the configuration manifold is $\mathbb{R}^{n+1}$, the movement of the particle is constrained to certain submanifold M of $\mathbb{R}^{n+1}$ and the Euler-Lagrange equation preserves its form for any chart of M. Moreover, if we restrict the velocities of the system to certain non-integrable distribution $N \subset TM$, we are dealing with *nonholonomic constraints* and Lagrange's equations no longer hold. Therefore, one more purpose in this thesis is to provide a generalization of the intrinsic EL equation to include nonholonomic constraints. In the following, we review part of the philosophy of lagrangian mechanics comparing it to newtonian mechanics, since the latter is more intuitive. Then, we proceed to explain the geometric structures used to achieve our purposes.

On the principles of mechanics

To determine the physical state of a mechanical system, one is required to provide the coordinates of position and velocity at each given time. Thus, a system of differential equations solves the problem:

$$\dot{x} = G(x, y), \quad \dot{y} = F(x, y), \tag{1}$$

where $(x, y) \in T\mathbb{R}^3$ are coordinates in the tangent bundle of the configuration space $\mathbb{R}^3$ and $G, F \in C^\infty(T\mathbb{R}^3, \mathbb{R}^3)$. Classical mechanics is best known formulated in Newton's laws of motion. The first and second laws provide us a notion of *force*, we cite them for reference:

1st law *A body remains at rest or in uniform motion unless acted upon by a force.*

2nd law *A body acted upon by a force moves in such a manner that the time rate change of momentum equals the force.*

The concept of force requires more elaboration, it becomes pertinent also in lagrangian mechanics. We will consider the mass of the body as 1, then, y is the momentum of the body. In this way, the second law is just the equation $\dot{y} = F(x, y)$, with $F(x, y)$ being the force. The first law is a condition for a system to be in equilibrium, we could rephrase it as: $\dot{x} = cnt \implies F = 0$, thus, relating the equations $\dot{x} = G(x, y)$ and $\dot{y} = F(x, y)$ for the case of equilibrium. In fact, for this case we can write $\ddot{x} = \dot{y} = F(x, y) = 0$, which, the newtonian formulation uses to extend the *equilibrium condition* to include dynamical systems ($\ddot{x} \neq 0$), arriving at the popular equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F(x, y) = \textit{force}. \tag{2}$$

Therefore, newtonian mechanics is formulated in second order differential equations (SODEs), from which we get positions and velocities $(x, y = \dot{x}) \in T\mathbb{R}^3$. As vector fields $\mathscr{X} \in \mathfrak{X}(T\mathbb{R}^3)$ on $T\mathbb{R}^3$, equations 1 are of the form:

$$\mathscr{X} = y^i \frac{\partial}{\partial x^i} + F^i(x, y)\frac{\partial}{\partial y^i}, \qquad i = 1, 2, 3. \tag{3}$$

In general, vector fields on tangent bundles $T\mathbb{R}^n$ that satisfy $\dot{x} = y$ are called **semisprays** on $T\mathbb{R}^n$. Semisprays are of special interest in analytical mechanics since they represent SODEs, e.g., in this case $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - F^i(x, y) = 0$. Hence, the use of semisprays will be recurrent along the thesis.

From equilibrium to dynamics The method of extending an equilibrium principle to set a dynamical formulation is found also in the *D'Alembert principle*. We elaborate on this. The conditions for a system to be in static equilibrium, i.e., total force and torque both equals zero, are replaced by *the principle of virtual work* which allows taking *constraint forces* into consideration. As mentioned above, constraints (holonomic) are submanifolds in where particles are restricted to move, e.g., a pendulum is a particle in $\mathbb{R}^2$ constrained to a circle $\mathbb{S}^1$.

Recall the definition of *work done by a force F* through a path $\gamma : [0, 1] \to \mathbb{R}^n$:

$$W[\gamma] = \int_\gamma F \cdot \mathrm{d}x = \textit{work}.$$

In equilibrium, the total force equals zero and as such, exerts no work on the particle. In this case, $F \cdot \mathrm{d}x = 0$. Constraints forces are forces produced on the particle to keep it in the constraint. An important *physical* assumption is that constraint forces *do not work* under any displacement, meaning, the movement is generated only by *non-constraint* forces. Thus, under equilibrium, the principle of virtual work asserts that no work is done on the system through any displacement in the constraint. To enunciate the principle, the following definitions are useful.

Definition 0.0.1. Let $x_0 \in \mathbb{R}^{n+1}$ represent the position of a system and $M \subset \mathbb{R}^{n+1}$ a constraint submanifold. A **virtual displacement** δx is a displacement $\delta x = x_1 - x_0$ such that $x_1 \in M$ and $\delta x \in T_{x_0}M$.

Definition 0.0.2. Given a force F and a virtual displacement $\delta x \in T_{x_0}M$, the **virtual work** is defined as

$$\delta W := F \cdot \delta x.$$

The adjective *virtual* is to differentiate virtual displacements from displacements corresponding to a translation in time of the state of the system.

Principle of virtual work *Let $x \in \mathbb{R}^n$ denote the position of a system and F a vector field representing the total force applied to it (including constraint forces). If the system is in equilibrium, then the virtual work done by the force is zero, i.e.,*

$$\dot{x} = cnt \quad \Longrightarrow \quad F \cdot \delta x = 0. \tag{4}$$

□

In general, if the system is not in equilibrium, then $F = \dot{y}$. D'Alembert uses this equation to extend the principle of virtual work for static systems to include dynamical systems:

D'Alembert's principle *Let $(x, y) \in T\mathbb{R}^n$ represent the state of a system, F the force applied and $M \subset \mathbb{R}^{n+1}$ a constraint submanifold. Then:*

$$\left(F - \dot{y}\right) \cdot \delta x = 0. \tag{5}$$

□

Euler-Lagrange equations

If we consider the submanifold $M \subset \mathbb{R}^{n+1}$ as the configuration manifold, we regard TM as the phase space and we determine the state with points in TM. Recall the definition of *kinetic energy* and *conservative force*, $T = \frac{1}{2}\|y\|^2$ and $F = -\nabla V$, respectively. From the d'Alembert's principle, we can deduce a new set of equations modeling mechanical systems, the so-called *Euler-Lagrange equations* ([Goldstein et al., 2002]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0, \qquad i = 1, \ldots, n, \quad (x, y) \in TM, \tag{6}$$

where $L = T - V$, and thus, replacing Newton's second law. The function $L \in C^\infty(TM)$ is called *lagrangian function*. Euler-Lagrange equations are also derived from the *Hamilton's principle*, which states that over all *admissible paths* (i.e., through virtual displacements) of a particle from a point to another in M, the path $\gamma : [0,1] \to M$ taken by the particle is such that extremizes:

$$\int_0^1 L(\gamma(t), \dot{\gamma}(t))\mathrm{d}t, \tag{7}$$

where $L : TM \to \mathbb{R}$ is a *lagrangian function*. This method is used in Chapter 2 to get equation 6 in invariant form. More specifically, we will construct a symplectic form $\Omega_L \in \Omega^2(TM)$ such that a particular hamiltonian field on the symplectic manifold (TM, Ω_L) has as integral curves the curves $(\gamma, \dot{\gamma}) : [0,1] \to TM$ that extremizes 7. This is resembles to the *hamiltonian formulation* of mechanics.

Hamilton equations

The hamiltonian formulation of mechanics is an application of *symplectic geometry*. Here, instead of describing the state of a system as points in the tangent bundle TM we describe it as points in the cotangent bundle T^*M, and the vector fields on T^*M whose integral lines are the states are called *hamiltonian vector fields*. The cotangent bundle carries a canonical symplectic form $\omega \in \Omega^2(T^*M)$. A hamiltonian vector field $\mathscr{X} \in \mathfrak{X}(T^*M)$ is one for which there exists a function $H \in C^\infty(T^*M)$ such that:

$$\imath_{\mathscr{X}}\omega = -\mathrm{d}H. \tag{8}$$

Due to *Darboux theorem*, in local coordinates $(x, p) \in T^*M$ hamiltonian vector fields are in the form of *Hamilton's equations*:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \quad \text{or} \quad \mathscr{X} = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}. \tag{9}$$

Take for example, $H = \frac{1}{2}p \cdot p + V(x) \in C^\infty(T^*\mathbb{R}^3)$, then

$$\mathscr{X} = p^i \frac{\partial}{\partial x^i} - \frac{\partial V}{\partial x^i}\frac{\partial}{\partial p^i}.$$

In addition, if we suppose that the momentum is just the mass times the velocity of the particle, then this vector field is the same semispray we gave in (3) with $F^i = -\partial V / \partial x^i$.

The tangent bundle doesn't have naturally a symplectic structure associated. Nevertheless, a lagrangian function $L : TM \to \mathbb{R}$ can be manipulated to define a suitable (pre)symplectic form on TM. Hence, the lagrangian formulation can be seen as an application of a geometric structure, where the objects defining the structure are the configuration manifold M and a lagrangian function $L : TM \to \mathbb{R}$.

Lagrange geometry

Lagrange geometry studies properties emerging from the two fundamental objects in problems of lagrangian mechanics: the configuration manifold and the lagrangian function. Let M be a manifold and $L : TM \to \mathbb{R}$ a lagrangian function, the pair (M, L) is called a **Lagrange space**. Our priority is to define a symplectic form $\Omega_L \in \Omega^2(TM)$ (in general presymplectic but we focus only on the symplectic case) such that hamiltonian vector fields $\mathscr{X} \in \mathfrak{X}(TM)$ of a particular hamiltonian function $E_L : TM \to \mathbb{R}$,

$$\imath_{\mathscr{X}}\Omega_L = -\mathrm{d}E_L, \tag{10}$$

are semisprays such that their integral curves project under the natural projection $\tau_M : TM \to M$ to the solutions of the Euler-Lagrange equation. Consider the function $\mathbb{J}$ locally given by:

$$\mathbb{J} : TTM \to TTM \; : \; (x, y; A, B) \to (x, y; 0, A).$$

Then, the 2−form

$$\Omega_L = \mathrm{d}\,(\mathrm{d}L \circ \mathbb{J})\,, \tag{11}$$

satisfies our requirements, as proved in Chapter 2. We remark that the 2−form Ω_L is in general presymplectic, and it is non-degenerated for a special type of lagrangians called *regulars*. We see that Equation (10) in the form $\imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L = 0$ is locally the EL equation.

As mentioned, then EL equation no longer holds in presence of nonholonomic constraints. In Chapter 3 we provide a way of incorporating external forces to the system as a special kind of differential 1−forms, the so-called *semibasic 1−forms*, and we define *mechanical systems* as triples (M, L, ω) where (M, L) is a Lagrange space and ω a force. Then, the *Lagrange-d'Alembert principle* follows the same philosophy we have been carrying around: *use an equilibrium principle to set a dynamical formulation*. The equilibrium principle in this case is that of forces annihilating under addition. We prove that $\Lambda(\mathscr{X}) := \imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L$ where $\mathscr{X}$ is any semispray, is always a semibasic 1−form, thus, a force. If $-\omega \in \Omega^1(TM)$ is an external force, then the Lagrange-d'Alembert principle says:

Lagrange-d'Alembert principle The *trajectories* $\hat{\gamma} : [0, 1] \to TM$ of a mechanical system (M, L, ω) are integral curves of the semispray $\mathscr{X} \in \mathfrak{X}_{\mathrm{ss}}(TM)$ such that

$$\Lambda(\mathscr{X}) - \omega = 0. \tag{12}$$

After this fundamental principle, we are ready to incorporate nonholonomic constraints. We define in Chapter 5 nonholonomic constraints as submanifolds $N \subset TM$. Providing a *physical acceptable* solution to a mechanical system with constraints is to find the *reaction force* and the correspondent *admissible semispray*, i.e., the force ω such that the semispray $\Lambda(\mathscr{X}) = \omega$ is tangent to N (it is admissible in the sense that its integral curves are in N).

In this way, we have set a fine notion of a constraint in lagrangian mechanics, i.e., we have restrict the motion and velocities to certain $N \subset TM$ and we found the semispray describing the trajectories of the mechanical system (a SODE, in accordance with the laws of physics).

Thesis structure

We now summarize the contents of each chapter of this work:

Ch. 1 • The second tangent bundle Working on the tangent bundle of a manifold and with vector fields on it, requires a study of the *second tangent* ([Antonelli, 2003]). This chapter is considered preliminary, it contains mathematical objects associated to TTM and properties of them that will be used along the thesis, e.g., the *vertical endomorphism, the liouville vector field, semisprays, semi-basic forms* ([Grifone, 1972] and [Szilasi et al., 2013]).

Ch. 2 • Lagrangian mechanics - Interlude In this chapter we focus on the derivation of the Euler-Lagrange equation. The EL equation is derived in an invariant way (without the use of coordinates) and a local expression is shown to be the classical EL equation ([Nester, 1988]).

Ch. 3 • Lagrange geometry Since the phase space of lagrangian mechanics is the tangent bundle, it is well suited in the formalism of the second tangent. In this chapter we study properties of lagrangian functions and the corresponding symplectic structures associated. We introduce the notions of *forces* and *accelerations* in the formalism in order to look for physical acceptable solutions to mechanical systems ([De León and Rodrigues, 2011] and [Grifone and Mehdi, 1999]). In this view, we finish with the Lagrange-d'Alembert principle ([Marsden and Ratiu, 1995]) .

Ch. 4 • Connections Ehresmann connections define a subbundle of the second tangent bundle complementary to the kernel of the tangent map of the tangent bundle projection (the vertical subbundle), called *horizontal subbundle*. Connections are relevant in lagrangian mechanics since we can find horizontal subbundles that are lagrangian with respect to the lagrangian $2-$form. Moreover, it is posible to define *geodesics* in terms of connections ([Szilasi et al., 2013] and [Miron and Anastasiei, 2012]) which allow us to verify the variational nature of lagrangian mechanics.

5.• Constraints Constraints restrict the movement of a given mechanical system. We are interested on nonholonomic constraints, those constraints that restrict the movement to a nonintegralbe distribution $N \subset TM$. It turns out, that we can a associate with N a set of forces that modify the semispray in the Lagrangrange-d'Alembert principle, in order to maintain the integral curves of the modified semispray in N ([Vershik and Faddeev, 1995] and [Grifone and Mehdi, 1999]). The new semispray is called *admissible semispray*. We workout the example of a free particle on a nonholonomic constraint and the vertical disk rolling without slipping ([Mladenova et al., 2014]).

Chapter 1

The second tangent bundle

Along the thesis, the vector bundle $\tau_M : TM \to M$ plays a fundamental role for endowing TM with a geometric structure, needed to study geometrically lagrangian mechanics with nonholonomic constraints. In this chapter we work with the vector bundle $\tau_{TM} : TTM \to TM$, the so-called *second tangent bundle*. The existence of the *vertical endomorphism* $\mathbb{J} : TTM \to TTM$ (Definition 1.2.3), allows several useful constructions in the second tangent, one of which was already illustrated in the introduction (the presymplectic form Ω_L, Equation (11)). Another application of the vertical endomorphism is the characterization of *semisprays* (Definition 1.3.1), these are vector fields on TM whose integral curves (say $\hat{\gamma}$) are at every time the velocity vectors of its projection under τ_M (this is, $\hat{\gamma}(t) = \mathrm{d}/\mathrm{d}t(\tau_M \circ \hat{\gamma}(t))$).

In the first section, *Two vector bundles*, we present another vector bundle $\tau_{M*} : TTM \to TM$ and its relation with τ_{TM}. The section *Vertical subbundle* includes the formal definition of the vertical endomorphism. In the section *Second order differential equations* we show how the previously elucidated properties of the second tangent allow a simple and convenient definition of semisprays. The last section is devoted to *semibasic forms*, whose relevance is seen in subsequent chapters as they allow us to define mathematically a notion of *force*. All these constructions are found in the books [Antonelli, 2003], [Grifone, 1972] and [Szilasi et al., 2013].

1.1 Two vector bundles $TTM \to TM$

We already have the vector bundles, $\tau_M : TM \to M$ and $\tau_{TM} : TTM \to TM$. The differential of the tangent bundle $\tau_M : TM \to M$, induces another vector bundle $\tau_{M*} : TTM \to TM$, so that, together with $\tau_{TM} : TTM \to TM$ have two vector bundles on TM. For a local coordinate chart (U, ϕ) of M, $U \subset \mathbb{R}^n$, we write elements in $TU \cong U \times \mathbb{R}^n$ as $(x, y) \in TU$, and elements in $TTU \cong (U \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$ as $(x, y; A, B)$. The coordinates (x, y) are called *associated coordinates with the base coordinates*, induced by the chart (U,ϕ). We are using the following abuse of notation when dealing with local coordinates: $(x, y) \in TM$, $(x, y; A, B) \in TTM$. Then, each bundle projection is locally

$$\begin{aligned} \tau_{TM} &: TTM \to TM : (x, y; A, B) \mapsto (x, y) \quad \text{and} \\ \tau_{M*} &: TTM \to TM : (x, y; A, B) \mapsto (x, A). \end{aligned}$$

Now, we present two important functions for the construction of the vertical endomorphism. First, we have the following commutative diagram

$$\begin{array}{ccc} TTM & \xrightarrow[\tau_{M*}]{} & TM \\ {\scriptstyle \tau_{TM}}\downarrow & & \downarrow{\scriptstyle \tau_M} \\ TM & \xrightarrow[\tau_M]{} & M \end{array}$$

and the exact sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\ \boldsymbol{i}\ } TTM \xrightarrow{\ \boldsymbol{j}\ } TM \times_M TM \longrightarrow 0. \tag{1.1}$$

Here, $TM \times_M TM := \{(v, w) \in TM \times TM \mid \tau_M(v) = \tau_M(w)\}$ and

$$\boldsymbol{i}(v, w) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v + tw), \qquad \text{and} \qquad \boldsymbol{j} := (\tau_{TM}, \tau_{M*}). \tag{1.2}$$

The functions (1.2) are used to define the vertical endomorphism. Now, elements in $TM \times_M TM$ are locally of the form $((x, y), (x, z))$. Thus,

$$\boldsymbol{i}((x, y), (x, z)) = (x, y, 0, z), \quad \text{and} \quad \boldsymbol{j}(x, y, A, B) = ((x, y), (x, A)). \tag{1.3}$$

Example 1.1.1. Consider de circle $\mathbb{S}^1$, then $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ and $TT\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. If $(\theta, y; A, B) \in TT\mathbb{S}^1$, then $\tau_{TM}(\theta, y; A, B) = (\theta, y)$ and $\tau_{M*}(\theta, y; A, B) = (\theta, A)$.

1.2 Vertical subbundle

We provide a special subbundle of TTM, the *vertical subbundle*. Note in the definition of the functions (1.2), that the codomain of **j** coincides with the domain of **i**. The composition of these functions is called *vertical endomorphism* and this endomorphism provides another characterization of the vertical subbundle (for this particular bundle) as both the kernel and the image the vertical endomorphism. We begin with the usual definition.

Definition 1.2.1. Consider the vector bundle $\tau_{TM} : TTM \to TM$, the **vertical subbundle** $\mathbb{V} \subset TTM$ of $\tau_{TM} : TTM \to TM$ is defined as

$$\mathbb{V} := \ker(\tau_{M*} : TTM \to TM). \tag{1.4}$$

Vectors $w \in \mathbb{V}_v$, $v \in TM$, are called **vertical vectors** and vector fields $\mathscr{X} \in \Gamma(\mathbb{V})$ are called **vertical vector fields**.

It is convenient to have at hand a basis for each vertical subspace. In fact, we see in the following lemma that the partial derivatives of the fiber coordinates serve as basis for the vertical subspaces.

Lemma 1.2.1. *Let M be a $n-$dimensional manifold. Then, for every $v \in TM$, the vertical subspace $\mathbb{V}_v \subset T_vTM$ is an $n-$dimensional subspace. Moreover, if $(x, y) \in TM$ are local coordinates, with $x \in M$ such that $v \in T_xM$, then*

$$\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\} \tag{1.5}$$

is a basis of $\mathbb{V}_v$.

Proof. The linear map $(\tau_{M*})_v : T_vTM \to T_xM$ sends the basis $\left(\left(\frac{\partial}{\partial x^i}\right)_v, \left(\frac{\partial}{\partial y^i}\right)_v\right)$ of T_vTM to a basis of T_xM. Since $(\tau_{M*})_v$ is surjective we have that $\dim\ker(\tau_{M*})_v + \operatorname{rank}(\tau_{M*})_v = 2n$, hence $\dim \mathbb{V}_v = n$. The vectors $\frac{\partial}{\partial y^i}$ are vertical, for any $f \in C^\infty(M)$:

$$\tau_{M*}\left(\frac{\partial}{\partial y^i}\right)(f) = \frac{\partial}{\partial y^i}(f \circ \tau_M) = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i} = 0,$$

Therefore, the vectors $\frac{\partial}{\partial y^i}$ span $\mathbb{V}_v$. □

The following definition is precisely the isomorphism between vertical subspaces and tangent spaces, as is proved in Proposition 1.2.1.

Definition 1.2.2. Let $v \in T_xM$, we define the **vertical lift** of $w \in T_xM$ to v as the map

$$\mathrm{vl}_v : T_xM \to \mathbb{V}_v \ : \ w \mapsto \mathrm{vl}_v(w) := \boldsymbol{i}(v, w). \tag{1.6}$$

Proposition 1.2.1. *The tangent space T_xM is canonically and linearly isomorphic to the vertical subspace $\mathbb{V}_v$ via the vertical lift.*

Proof. Note from the definition of $\boldsymbol{i}$ that $\mathrm{vl}_v(w) = 0$ if and only if $w = 0$. □

Remark 1.2.1. Note that the vertical subbundle can be regarded equivalently as:

$$\mathbb{V} = \operatorname{im}\boldsymbol{i} = \ker \boldsymbol{j}. \tag{1.7}$$

Indeed, just note that elements $w \in \mathbb{V}$ are of the form $w = (x, y; 0_{TM}, A)$.

Given a smooth map $\varphi : M \to N$ between the manifolds M and N, we say that two vector fields $\mathscr{X} \in \mathfrak{X}(M)$ and $\mathscr{Y} \in \mathfrak{X}(N)$ are $\varphi-$related if $\varphi_* \circ \mathscr{X} = \mathscr{Y} \circ \varphi$. Moreover, if $\mathscr{X}_1, \mathscr{X}_2 \in \mathfrak{X}(M)$ are $\varphi-$related to $\mathscr{Y}_1, \mathscr{Y}_2 \in \mathfrak{X}(N)$, respectively, then $[\mathscr{X}_1, \mathscr{X}_2]$ is $\varphi-$related to $[\mathscr{Y}_1, \mathscr{Y}_2]$. The following lemma is a characterization of vertical vector fields in this terminology and in this way we prove that vertical vector fields are a Lie subalgebra of vector fields on TM.

Lemma 1.2.2. *A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is vertical if and only if $\mathscr{X}$ is τ_M-related to the zero section $0_{TM} \in \mathfrak{X}(M)$.*

Proof. Suppose $\mathscr{X} \in \mathfrak{X}(TM)$ is vertical, this is $\tau_{M*}(\mathscr{X}_y) = 0_{\tau_M(y)}$ for every $y \in TM$, or equivalently $\tau_{M*} \circ \mathscr{X} = 0_{TM} \circ \tau_M$, this is $\mathscr{X} \sim_{\tau_M} 0_{TM}$. □

Proposition 1.2.2. *Vertical vector fields $\Gamma(\mathbb{V})$ form a Lie subalgebra of $\mathfrak{X}(TM)$.*

Proof. Indeed, since $\mathscr{X}, \mathscr{Y} \in \Gamma(\mathbb{V})$ are both τ_M-related to 0_{TM}, $[\mathscr{X}, \mathscr{Y}]$ is τ_M-related to $[0_{TM}, 0_{TM}] = 0_{TM}$, i.e.,

$$0_{TM} \circ \tau_M = \tau_{M*}[\mathscr{X}, \mathscr{Y}],$$

and thus, $[\mathscr{X}, \mathscr{Y}]$ is vertical. □

1.2.1 Vertical endomorphism

In this subsection we exploit the diagram (1.1) and functions (1.2) to construct the vertical endomorphism. The existence of this map is particular of the tangent bundle and it is of special relevance along the thesis. For example, the definition of the presymplectic form on TM takes advantage of the vertical endomorphism as an intrinsic object to the tangent bundle to construct the 2−form only depending on the Lagrangian function.

Definition 1.2.3. The **vertical endomorphism**, also called **(canonical) almost-tangent structure** on TM is the bundle map $\mathbb{J} : TTM \to TTM$ (or $\mathbb{J} \in \Omega^1(TM, TTM)$) defined as the function $\boldsymbol{j} : TTM \to TM \times_M TM$ composed with $\boldsymbol{i} : TM \times_M TM \to TTM$:

$$\mathbb{J} := \boldsymbol{i} \circ \boldsymbol{j}. \tag{1.8}$$

In local coordinates $(x, y) \in TM$ we can write the vertical endomorphism as

$$\mathbb{J} = \mathrm{d}x^i \otimes \frac{\partial}{\partial y^i}, \tag{1.9}$$

and therefore, for the vertical endomorphism acting on a vector field $(x, y; A(x, y), B(x, y))$ is

$$\mathbb{J}(x, y; A(x, y), B(x, y)) = (x, y; 0, A(x, y)).$$

It readily follows from the local expression of $\mathbb{J}$ that

$$\mathbb{V} = \operatorname{im}\mathbb{J} = \ker\mathbb{J} \qquad \text{and} \qquad \mathbb{J}^2 = 0_{TTM}. \tag{1.10}$$

Hence, we have alternative characterizations of the vertical subbundle as the image and the kernel of the vertical subbundle.

An important property of the vertical endomorphism is the vanishing of the *Nijenhuis tensor* (or *Nijenhuis torsion*) of the vertical endomorphism. Given a vector-valued 1−form $Q \in \Omega^1(TM, TTM)$, the Nijenhuis torsion of Q is defined as the vector-valued 2−form $\mathcal{N}_Q \in \Omega^2(TM, TTM)$ given by

$$\mathcal{N}_Q = [Q, Q]_{FN}. \tag{1.11}$$

Here, $[\cdot,\cdot]_{FN}$ stands for the *Frölicher-Nijenhuis bracket* (see Appendix A and references therein).

Proposition 1.2.3. *The Nijenhuis tensor of the vertical endomorphism vanishes, i.e.,*

$$\mathcal{N}_{\mathbb{J}}(\mathscr{X},\mathscr{Y}) = 0,$$

for every $\mathscr{X},\mathscr{Y} \in \mathfrak{X}(TM)$.

Proof. From Proposition A.2.2 we have that the expression for the Nijenhuis bracket evaluated in vector fields $\mathscr{X},\mathscr{Y} \in \mathfrak{X}(TM)$ is

$$\mathcal{N}_{\mathbb{J}}(\mathscr{X},\mathscr{Y}) = [\mathbb{J}\mathscr{X},\mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathbb{J}\mathscr{X},\mathscr{Y}] - \mathbb{J}[\mathscr{X},\mathbb{J}\mathscr{Y}].$$

Since $\mathcal{N}_{\mathbb{J}}$ is a tensor it is sufficient to evaluate it in a local basis of vector fields, say $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\right\}$. Then, calculating each term in the basic elements $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^j}$ we have

$$\left[\mathbb{J}\frac{\partial}{\partial x^i},\mathbb{J}\frac{\partial}{\partial y^j}\right] = \left[\frac{\partial}{\partial y^i},0\right] = 0, \qquad \left[\mathbb{J}\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^j}\right] = \left[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right] = 0, \qquad \left[\frac{\partial}{\partial x^i},\mathbb{J}\frac{\partial}{\partial y^j}\right] = \left[\frac{\partial}{\partial x^i},0\right] = 0,$$

and similar for the remaining combinations of the basic elements. □

1.2.2 Liouville vector field and homogeneity

The following is the definition of the *Liouville vector field*. Together with the vertical endomorphism, the Liouville vector field is part of our collection of *canonical* objects ascribed to TTM. In fact, this vector field exists in every vector bundle since its construction only requires the $\boldsymbol{i}$ function. Nevertheless, in our context, the Liouville vector field is used, together with the vertical endomorphism, to characterize second order differential equations. Also, we introduce the notion of *homogeneity* of functions and tensor fields, a useful property in subsequent calculations. For example, the homogeneity property of the vertical endomorphism follows from the fact that its Lie derivative along the Liouville vector field equals minus the vertical endomorphism (Example 1.2.1).

Definition 1.2.4. The **Liouville vector field** $\mathscr{V} \in \mathfrak{X}(TM)$ is the vector field defined by:

$$\mathscr{V} : TM \to TTM \, : \, v \mapsto \boldsymbol{i}(v, v). \tag{1.12}$$

Locally, the expression for the Liouville vector field in $(x, y) \in TM$ is

$$\mathscr{V} = y^i \frac{\partial}{\partial y^i}. \tag{1.13}$$

Evidently, the Liouville vector field is vertical. For now on, the letter $\mathscr{V}$ is reserved for the Liouville vector field.

Definition 1.2.5. Define the **homothetic of ratio** λ, where $\lambda \in (0, \infty)$, as the function $h_\lambda : TM \to TM$ defined by

$$h_\lambda(x, y) = (x, \lambda y).$$

Remark 1.2.2. The set of all homothetics is a one-parameter group and the Liouville vector field has the group of homothetics as one-parameter group.

Definition 1.2.6. A function $f \in C^\infty(TM)$ is called **homogeneous of degree** r if

$$f \circ h_\lambda = \lambda^r f, \qquad \lambda > 0.$$

Proposition 1.2.4 (Euler type theorem for homogeneous functions)**.** *A function $f \in C^\infty(TM)$ is homogeneous of degree r if and only if the Lie derivative of f along f is equal to rf, i.e.,*

$$\mathcal{L}_{\mathscr{V}} f = rf.$$

In local coordinates $\frac{\partial f}{\partial y^i} y^i = rf$.

Definition 1.2.7. A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is called **homogeneous of degree** r if

$$\mathscr{X} \circ h_\lambda = \lambda^{r-1} (h_\lambda)_* \mathscr{X}, \qquad \lambda > 0.$$

Proposition 1.2.5 (Euler type theorem for homogeneous vector fields)**.** *A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is homogeneous of degree r if and only if*

$$\mathcal{L}_{\mathscr{V}} \mathscr{X} = (r-1)\mathscr{X}.$$

Remark 1.2.3. The Euler type theorem for homogeneous vector fields in local coordinates says that $\mathscr{X} = (x, y; A(x, y), B(x, y))$ is homogeneous of degree r if and only if $A(x, y)$ is homogeneous of degree $r - 1$ and $B(x, y)$ is homogeneous of degree r. Indeed,

$$\mathcal{L}_{\mathscr{V}}(\mathscr{X}) = y^j \frac{\partial A^i(x, y)}{\partial y^j} \frac{\partial}{\partial x^i} + \left(y^j \frac{\partial B^i(x, y)}{\partial y^j} - B^i(x, y) \right) \frac{\partial}{\partial y^i},$$

therefore, $\mathcal{L}_{\mathscr{V}}(\mathscr{X}) = (r-1)\mathscr{X}$ if and only if

$$y^j \frac{\partial A^i(x,y)}{\partial y^j} = (r-1)A^i(x,y), \quad \text{and} \quad y^j \frac{\partial B^i(x,y)}{\partial y^j} - B^i(x,y) = (r-1)B^i(x,y).$$

Definition 1.2.8. A tensor field T of $(1,s)-$type is homogeneous of degree r if

$$\mathcal{L}_{\mathscr{V}} T = (r-1)T.$$

Example 1.2.1.

a. The Liouville vector field $\mathscr{V}$ is homogeneous of degree 1.

b. The vertical endomorphism $\mathbb{J}$ is a $(1,1)-$type homogeneous tensor of degree 0, i.e.,

$$\mathcal{L}_{\mathscr{V}} \mathbb{J} = -\mathbb{J}.$$

Indeed, by the Leibniz identity we have that this is equivalent to

$$[\mathscr{V}, \mathbb{J}\mathscr{X}] - \mathbb{J}[\mathscr{V}, \mathscr{X}] = -\mathbb{J}\mathscr{X}, \quad \text{for every } \mathscr{X} \in \mathfrak{X}(TM),$$

which is immediate by evaluating in the vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$.

1.3 Second order differential equation

In this section we provide a convenient definition of second order differential equations (SODEs) on manifolds in terms of vector fields on the tangent bundle, by means of the notion of *semispray*. The importance of these vector fields was already emphasized in the Introduction of the thesis and in the introduction of this Chapter: their integral curves represent at each time the velocity of their projection under τ_M (Corollary 1.3.1), which gives these vector fields the character of SODEs.

Definition 1.3.1. A **semispray** or **second order differential equation** (SODE) $\mathscr{X}$ over M is a section of $\tau_{TM} : TTM \to TM$ such that

$$\mathbb{J}\mathscr{X} = \mathscr{V}. \tag{1.14}$$

The semispray $\mathscr{X} \in \mathfrak{X}(TM)$ is called **spray** if additionally satisfies:

$$[\mathscr{V}, \mathscr{X}] = \mathscr{X}, \tag{1.15}$$

i.e., $\mathscr{X}$ is an homogeneous vector field of order 2.

Remark 1.3.1. Note that if $\mathscr{X} \in \mathfrak{X}(TM)$ is a semispray, then for any vertical vector field $\mathscr{Y} \in \Gamma(\mathbb{V})$, the vector field $\mathscr{X} + \mathscr{Y}$ is also a semispray, and conversely. Therefore, the set of semisprays over M is an affine space modelled over the vertical vector fields on TTM.

The following proposition provides a characterization of semisprays which justifies the name SODE for semisprays (see Remark 1.3.2).

Proposition 1.3.1. *A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is a semispray if and only if for any local chart (x, y) of TM there exist functions $G^i \in C^\infty(TM)$ such that*

$$\mathscr{X} = (x, y; y, -2G(x, y)), \qquad \textit{where } G := (G^1, \ldots, G^n),\ G^i \in C^\infty(TM).$$

Moreover, $\mathscr{X} \in \mathfrak{X}(TM)$ is a spray if and only if:

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \textit{for any } \lambda > 0.$$

Proof. A vector field in a local chart (x, y) is of the form $\mathscr{X} = (x, y; A(x, y), B(x, y))$. From the condition for $\mathscr{X}$ to be a semispray, we have

$$\mathbb{J}(\mathscr{X}) = \mathbb{J}((x, y; A(x, y), B(x, y)) = (x, y; 0_{TM}, A(x, y)) = \mathscr{V}, \quad \text{where} \quad \mathscr{V} = (x, y; 0_{T_xM}, y).$$

Thus, $A^i(x, y) = y^i$. Choosing $G^i(x, y) = -(1/2)B^i(x, y)$ we get

$$\mathscr{X} = (x, y; y, -2G(x, y)).$$

The second part is due to the Euler type theorem for homogeneous vector fields. □

The functions $-2G^i(x, y)$ are called **local coefficients** of the semispray. In the following remark we elaborate on the interpretation of semisprays as second order differential equations.

Remark 1.3.2. The integral curves of $\mathscr{X}$ are solutions of the equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = y^i, \qquad \frac{\mathrm{d}y^i}{\mathrm{d}t} = -2G^i(x, y),$$

which, replacing y^i by $\mathrm{d}x^i/\mathrm{d}t$, gives the second order differential equations on M:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + 2G^i(x, y) = 0, \qquad i = 1, \ldots, n. \tag{1.16}$$

The -2 factor in the coefficients of the semisprays is relevant until Chapter 4, where we define *connections* determined by semisprays and the -2 simplifies the expression of the coefficients of the connection. The following proposition says that semisprays are sections of both bundles $\tau_{M*} : TTM \to TM$ and $\tau_{TM} : TTM \to TM$ simultaneously.

Proposition 1.3.2. *A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is a semispray if and only if*

$$\tau_{M*} \circ \mathscr{X} = \mathbb{I}_{TM},$$

i.e., $\mathscr{X}$ is a section of $\tau_{M*} : TTM \to TM$.

Proof. Write $\mathscr{X}$ in local coordinates, $\mathscr{X} = (x, y; A(x, y), B(x, y))$. Then,

$$\tau_{M*}(\mathscr{X}) = \tau_{M*}((x, y; A(x, y), B(x, y)) = (x, A(x, y)).$$

By noticing that $\mathscr{X}$ is a semispray if and only if $A(x, y) = y$, the result follows. □

As a final characterization of semisprays, we show what was mentioned at the beginning: semisprays are those vector fields on TM whose integral curves are at each time the velocities of their projection to M.

Corollary 1.3.1. *A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is a semispray if and only if each integral curve $\hat{\gamma} : [0, 1] \to M$ of $\mathscr{X}$ is satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau_M \circ \hat{\gamma}(t)\right) = \hat{\gamma}(t).$$

Proof. From the chain rule, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau_M \circ \hat{\gamma}(t)\right) = \tau_{M*} \circ \mathscr{X} \circ \hat{\gamma}(t) = \mathbb{I}_{TM} \circ \hat{\gamma}(t) = \hat{\gamma}(t)$. □

The following proposition shows a property of semisprays resulting from the nullity of the Nijenhuis tensor and the homogeneity of the vertical endomorphism.

Proposition 1.3.3. *Let $\mathscr{X} \in \mathfrak{X}(TM)$ be a semispray. Then, for any vector field $\mathscr{Y} \in \mathfrak{X}(TM)$, we have*

$$\mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] = -\mathbb{J}\mathscr{Y}. \tag{1.17}$$

Proof. From Proposition 1.2.3 we have that the Nijenhuis torsion of the vertical endomorphism is zero, this is,

$$\mathcal{N}_{\mathbb{J}}(\mathscr{X}, \mathscr{Y}) = [\mathbb{J}\mathscr{X}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathbb{J}\mathscr{X}, \mathscr{Y}] = 0.$$

If $\mathscr{X}$ is a semispray, then

$$[\mathscr{V}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathscr{V}, \mathscr{Y}] = 0.$$

From Example 1.2.1 we know that $\mathbb{J}$ is homogeneous of degree 0, i.e., $[\mathscr{V}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathscr{V}, \mathscr{Y}] = -\mathbb{J}\mathscr{Y}$. Therefore,

$$\mathbb{J}\mathscr{Y} - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] = 0.$$

□

1.4 Semibasic forms

Semibasic forms are a particular kind of differential forms essential for the thesis objectives, since $1-$forms of this kind represent *forces* in *mechanical systems*. Furthermore, we see that semibasic $1-$forms are in bijection with vertical vector fields (Corollary 3.2.3), and interpreting vertical vector fields as *accelerations* we have a correspondence between forces and accelerations.

In the following definition we use the adjoint of the map $\mathbb{J} : TTM \to TTM$ as the natural extension $\mathbb{J}^* : \bigwedge^\bullet T^*TM \to \bigwedge^\bullet T^*TM$.

Definition 1.4.1. A **semibasic form** on TM is a form $\omega \in \Omega^\bullet(TM)$ such that $\omega \in \operatorname{Im} \mathbb{J}^*$.

Remark 1.4.1. Since semibasic $1-$forms are given by $\omega = \mathbb{J}^*\alpha$ for some $\alpha \in \Omega^1(TM)$, it annihilates vertical vector fields, i.e.,

$$\omega(\mathscr{X}) = \mathbb{J}^*\alpha(\mathscr{X}) = \alpha \circ \mathbb{J}(\mathscr{X}) = 0, \quad \text{for any } \mathscr{X} \in \Gamma(\mathbb{V}).$$

In local coordinates $(x, y) \in TM$, semibasic $1-$forms are spanned by $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ and semibasic $k-$forms by

$$\left\{\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \quad | \quad 1 \le i_1 < \cdots < i_k \le n\right\}.$$

The following is a useful characterization of semibasic $1-$forms that complements the observations of the previous remark. Denote by $\Omega^1_{\mathrm{sb}}(TM)$ the set of semibasic $1-$forms.

Proposition 1.4.1. *Let $\omega \in \Omega^1(TM)$ be a $1-$form on TM, then ω is a semibasic form if and only if it annihilates vertical vector fields.*

Proof. By definition, $\Omega^1_{\mathrm{sb}}(TM) = \operatorname{Im} \mathbb{J}^* = (\ker \mathbb{J})^\circ = \mathbb{V}^\circ$, where $\mathbb{V}^\circ$ denotes the annihilator of $\mathbb{V}$. □

Definition 1.4.2. A semibasic $1-$form $\omega \in \Omega^1(TM)$ is called **basic** if there exists a $1-$form $\alpha \in \Omega^1(M)$ on M such that

$$\omega = \tau_M^*\alpha. \tag{1.18}$$

Proposition 1.4.2. *Let $\omega \in \Omega^1(TM)$ be a $1-$form on TM, then ω is a basic $1-$form if and only if*

$$\imath_{\mathscr{X}}\omega = 0 \quad \textit{and} \quad \mathcal{L}_{\mathscr{X}}\omega = 0, \quad \textit{for any } \mathscr{X} \in \Gamma(\mathbb{V}). \tag{1.19}$$

Proof. Let ω be a semibasic $1-$form, then we can write it locally as $\omega = \omega_i(x, y)\mathrm{d}x^i$. Since the bundle projection is $(x, y) \mapsto x$, just observe that if ω is basic, $\omega = \omega_i(x)\mathrm{d}x^i$. □

Chapter 2

Lagrangian mechanics - Interlude

The purpose of this chapter is to derive the *Euler-Lagrange (EL) equation* on its invariant form (Theorem 2.2.1) using the constructions of the previous chapter. The idea is to define a *suitable* (pre)symplectic form Ω_L on the tangent bundle TM which only depends on the configuration manifold M and the lagrangian $L : TM \to \mathbb{R}$; by *suitable* we mean that we construct it so that a hamiltonian vector field of the (pre)symplectic manifold (TM, Ω_L) represents the EL equation as a SODE (i.e., a semispray).

In the first section, *Lagrangian forms*, we define the presymplectic structure Ω_L that serves for our purposes. The second section, *Variational calculus*, contains the main result (the Theorem 2.2.1). Since the EL equation is of variational nature, we provide a framework for variational calculus. With this, we show that the solutions of the EL are curves extremizing the functional $\int L$ in the *fundamental problem*. Moreover, this result also holds for the intrinsic EL equation as we see in the Theorem , in this case curves extremizing the functional $\int L$ are the integral curves of a hamiltonian $\imath_{\mathscr{X}} \Omega_L = -\mathrm{d}E_L$. We mainly follow [Nester, 1988].

2.1 Lagrangian forms

The tangent bundle TM doesn't carries a natural symplectic structure. In this section we define a presymplectic form on TM, the so-called *lagrangian* $2-$form. Then, we show the case for when the lagrangian $2-$form is non-degenerate, i.e., symplectic. Lagrangian functions giving place to symplectic lagrangian $2-$forms are called *regular lagrangians*.

Definition 2.1.1. Let $L : TM \to \mathbb{R}$ be a lagrangian function and $\mathbb{J} : TTM \to TTM$ the vertical endomorphism. The differential forms,

$$\Theta_L := \mathcal{L}_{\mathbb{J}} L = \mathrm{d}L \circ \mathbb{J} \in \Omega^1(TM) \tag{2.1}$$

$$\text{and} \qquad \Omega_L := \mathrm{d}(\mathcal{L}_{\mathbb{J}} L) = \mathrm{d}(\mathrm{d}L \circ \mathbb{J}) \in \Omega^2(TM), \tag{2.2}$$

are called **lagrangian** 1−**form** and **lagrangian** 2−**form**, respectively.

Symplectic lagrangian 2−**form** Note that the lagrangian 2−form Ω_L is closed since by definition is exact, therefore, (TM, Ω_L) is a presymplectic manifold. We shall describe the case when it is symplectic. In local coordinates $(x, y) \in TM$, we have that

$$\Omega_L = \mathrm{d}\Theta_L = \mathrm{d}\left(\frac{\partial L}{\partial y^i}\mathrm{d}x^i\right) = \frac{\partial^2 L}{\partial x^j \partial y^i}\mathrm{d}x^j \wedge \mathrm{d}x^i + \frac{\partial^2 L}{\partial y^j \partial y^i}\mathrm{d}y^j \wedge \mathrm{d}x^i. \tag{2.3}$$

Therefore, we see that the associated matrix to Ω_L takes the form

$$[\Omega_L] = \begin{bmatrix} \left[\frac{\partial^2 L}{\partial x^i \partial y^j} - \frac{\partial^2 L}{\partial x^j \partial y^i}\right] & -\left[\frac{\partial^2 L}{\partial y^j \partial y^i}\right] \\ \left[\frac{\partial^2 L}{\partial y^j \partial y^i}\right] & 0 \end{bmatrix}. \tag{2.4}$$

From this matrix we see that the 2−form Ω_L is non-degenerate if and only if the matrix $\left[\partial^2 L/\partial y^j \partial y^i\right]$ is invertible.Whether Ω_L is symplectic or presymplectic depends only on the lagrangian function. For those lagrangians inducing a symplectic 2−form by the construction we gave, there is a special name.

Definition 2.1.2. A lagrangian function $L : TM \to \mathbb{R}$ is said to be **regular** if the lagrangian 2−form $\Omega_L = \mathcal{L}_{\mathbb{J}} L$ is a symplectic form on TM.

2.1.1 A hamiltonian vector field on (TM, Ω_L)

We describe the hamiltonian vector field associated with a particular hamiltonian function, the so-called *energy function*. We see that, in the special case when Ω_L is symplectic, this hamiltonian vector field locally represents the EL equation. In the symplectic case it can be shown, moreover, that this hamiltonian vector field is a semispray. We prove this fact in local coordinates in this section and the intrinsic prove is given in Chapter 3.

Definition 2.1.3. Let $L : TM \to \mathbb{R}$ be a lagrangian function and $\mathscr{V} \in \mathfrak{X}(TM)$ the Liouville vector field. The **energy function** $E_L : TM \to \mathbb{R}$ associated with the lagrangian is defined as

$$E_L := \mathcal{L}_{\mathscr{V}} L - L. \tag{2.5}$$

Now, a hamiltonian vector field $\mathscr{X} \in \mathfrak{X}(TM)$ associated with E_L is such that $\imath_{\mathscr{X}}\Omega_L = -\mathrm{d}E_L$. Note that since the 2−form is in general degenerate, the hamiltonian vector field might not exists or might not be unique for each hamiltonian. In the following we suppose that at least one hamiltonian vector field exists for the energy function.

Proposition 2.1.1. *Let $L : TM \to \mathbb{R}$ be a lagrangian function and consider the presymplectic manifold (M, Ω_L). Then, vector fields $\mathscr{X} \in \mathfrak{X}(TM)$ such that*

$$\imath_{\mathscr{X}}\Omega_L = -\mathrm{d}E_L, \tag{2.6}$$

locally seen as $\mathscr{X} = (x, y; A(x, y), B(x, y))$, satisfy

$$\frac{\partial^2 L}{\partial x^i \partial y^j} A^i(x,y) + \frac{\partial^2 L}{\partial x^j \partial y^i}\left(y^i - A^i(x,y)\right) + \frac{\partial^2 L}{\partial y^i \partial y^j} B^i(x,y) - \frac{\partial L}{\partial x^j} = 0 \tag{2.7}$$

$$\text{and} \qquad \frac{\partial^2 L}{\partial y^j \partial y^i}(A^i(x,y) - y^i) = 0. \tag{2.8}$$

Proof. We just calculate each side of Equation (2.6),

$$\begin{aligned}
\imath_{\mathscr{X}}\Omega_L &= \imath_{\mathscr{X}}\left(\frac{\partial^2 L}{\partial x^j \partial y^i}\mathrm{d}x^j \wedge \mathrm{d}x^i + \frac{\partial^2 L}{\partial y^j \partial y^i}\mathrm{d}y^j \wedge \mathrm{d}x^i\right) \\
&= \frac{\partial^2 L}{\partial x^j \partial y^i}\left(A^j \mathrm{d}x^i - A^i \mathrm{d}x^j\right) + \frac{\partial^2 L}{\partial y^j \partial y^i}\left(B^j \mathrm{d}x^i - A^i \mathrm{d}y^j\right) \\
&= \left(\left(\frac{\partial^2 L}{\partial x^i \partial y^j} - \frac{\partial^2 L}{\partial x^j \partial y^i}\right)A^i + \frac{\partial^2 L}{\partial y^i \partial y^j}B^i\right)\mathrm{d}x^j - \frac{\partial^2 L}{\partial y^j \partial y^i}A^i \mathrm{d}y^j
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}E_L &= \mathrm{d}\left(y^i \frac{\partial L}{\partial y^i} - L\right) \\
&= \frac{\partial^2 L}{\partial x^j \partial y^i} y^i \mathrm{d}x^j + \frac{\partial^2 L}{\partial y^j \partial y^i} y^i \mathrm{d}y^j + \frac{\partial L}{\partial y^i}\mathrm{d}y^i - \frac{\partial L}{\partial x^i}\mathrm{d}x^i - \frac{\partial L}{\partial y^i}\mathrm{d}y^i \\
&= \left(\frac{\partial^2 L}{\partial x^j \partial y^i} y^i - \frac{\partial L}{\partial x^j}\right)\mathrm{d}x^j + \frac{\partial^2 L}{\partial y^j \partial y^i} y^i \mathrm{d}y^j.
\end{aligned}$$

Therefore, we have

$$\left(\frac{\partial^2 L}{\partial x^i \partial y^j} - \frac{\partial^2 L}{\partial x^j \partial y^i}\right)A^i + \frac{\partial^2 L}{\partial y^i \partial y^j}B^i = -\frac{\partial^2 L}{\partial x^j \partial y^i} y^i + \frac{\partial L}{\partial x^j}, \tag{2.9}$$

$$\frac{\partial^2 L}{\partial y^j \partial y^i}A^i = \frac{\partial^2 L}{\partial y^j \partial y^i} y^i. \tag{2.10}$$

□

The following corollary and its remark show how we can regard $\iota_{\mathscr{X}}\Omega_L + \mathrm{d}E_L = 0$ in coordinates as the EL equation for the particular case when Ω_L is non-degenerate. In the next section we prove this fact departing from a variational principle and in a coordinate-free manner. Furthermore, we can see locally that this vector field is a semispray.

Corollary 2.1.1. *If the lagrangian is regular, then equations* (2.7) *and* (2.8) *reduce to*

$$\frac{\partial^2 L}{\partial y^i \partial y^j} B^i(x,y) + \frac{\partial^2 L}{\partial x^i \partial y^j} y^i - \frac{\partial L}{\partial x^j} = 0. \tag{2.11}$$

Proof. For the lagrangian L being regular is equivalent to $[\partial^2 L/\partial y^j \partial y^i]$ being invertible, which implies in (2.8)that $A^i(x,y) = y^i$ and the result follows. □

Remark 2.1.1. For a regular Lagrangian, *the hamiltonian vector field associated to the energy E_L is a semispray $\mathscr{X} = (x,y;y,B(x,y))$* since $A(x,y) = y$, therefore, $\dot{x} = y$. The intrinsic proof of the semispray property of this vector field is given in Proposition 3.1.3. Moreover, if we consider the EL equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j} - \frac{\partial L}{\partial x^j} = 0, \tag{2.12}$$

then we can see that it is exactly the Equation (2.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j} - \frac{\partial L}{\partial x^j} = \frac{\partial^2 L}{\partial y^i \partial y^j} \dot{y}^i(x,y) + \frac{\partial^2 L}{\partial x^i \partial y^j} \dot{x}^i - \frac{\partial L}{\partial x^j} = 0, \tag{2.13}$$

since $\dot{y} = B(x,y)$.

2.2 Variational calculus

In this section we review some variational calculus. We enunciate the *fundamental problem of variational calculus* from where it is derived the Euler-Lagrange (EL) equation in a classical manner by means of local coordinates. The subsection *Intrinsic Euler-Lagrange equation* dedicates a few paragraphs to translate the classical notion of *virtual displacement* to the language of vector fields on TM and then, with this construction, the coordinate-free (intrinsic) EL equation is derived.

Recall from the introduction the definition of virtual displacement: in a surface $M \subset \mathbb{R}^n$ a **virtual displacement** is a displacement $\delta x = x_1 - x_0$ such that $x_1 \in \Delta$ and $\delta x \in T_{x_0}\Delta$. Note that the following definition generalize the previous one for the case of $\gamma(t) = x_0$ for all t and arbitrary *variations*.

Definition 2.2.1. Let M be a manifold, and $\gamma : [0,1] \to M$ a smooth curve. A **variation** of γ is a C^2 function $\vartheta : [0,1] \times (-\varepsilon, \varepsilon) \to M$ such that

a. $\vartheta(t,0) = \gamma(t)$, for all $t \in [0,1]$, and

b. $\vartheta(0,u) = \gamma(0)$ and $\vartheta(1,u) = \gamma(1)$ for all $u \in (-\varepsilon,\varepsilon)$.

The $\vartheta-$**virtual displacement** of $\gamma : [0,1] \to M$ at t is defined by

$$\delta\gamma(t) := \left.\frac{\partial}{\partial u}\right|_{u=0} \vartheta(t,u) \in T_{\gamma(t)}M.$$

Notation We simplify the notation for velocities of a given curve γ with $\hat{\gamma} : [0,1] \to TM \; : \; t \mapsto \hat{\gamma}(t) := \big(\gamma(t), \dot{\gamma}(t)\big)$. Then, $\hat{\gamma}$ is a curve in TM. Given a variation ϑ of γ, we can define a variation of $\hat{\gamma}$:

$$\hat{\vartheta} : [0,1] \times (-\varepsilon,\varepsilon) \to TM \; : \; (t,u) \mapsto \hat{\vartheta}(t,u) := \left(\vartheta(t,u), \frac{\partial}{\partial t}\vartheta(t,u)\right).$$

The $\hat{\vartheta}-$virtual displacement at t is, therefore,

$$\delta\hat{\gamma}(t) = \left(\left.\frac{\partial}{\partial u}\right|_{u=0} \vartheta(t,u), \left.\frac{\partial^2}{\partial u \partial t}\right|_{u=0} \vartheta(t,u) \right) \in T_{\hat{\gamma}(t)}TM.$$

In order to extremize functionals we need to derivate in a space of curves. The family of curves given in variations serves for this purpose and the method is to derive along the variation parameter. First, we see how to define the variations of functions along curves and then we show the use of the *variational derivative* in the fundamental problem.

Definition 2.2.2. The $\vartheta-$**variation of a smooth function** $L : TM \to \mathbb{R}$ along the curve γ is defined by the Lie derivative of L along $\delta\hat{\gamma}$:

$$\delta L[\gamma] := \mathcal{L}_{\delta\hat{\gamma}} L = \mathrm{d}L\big(\delta\hat{\gamma}\big).$$

The following statement is considered the *fundamental problem of variational calculus*, from where the Euler-Lagrange equation is deduced.

The fundamental problem *Given two points $x_0,\ x_1 \in M$ in a manifold M, find a curve $\gamma : [0,1] \to M$, with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, such that extremizes* ($\mathscr{P}1$)

$$I[\gamma] = \int_0^1 L(\hat{\gamma}(t))\mathrm{d}t. \tag{2.14}$$

The fundamental problem is referred as *problem* $\mathscr{P}1$ and I is called the **action functional**.

Solution of $\mathscr{P}1$. The well known method for solving it is by evaluating the action functional on arbitrary variations $\vartheta = \vartheta(t,u)$ of γ, taking the *variational derivative* of the action functional,

$$\delta I[\gamma] := \left.\frac{\partial}{\partial u}\right|_{u=0} I[\vartheta(t,u)] = \int_0^1 \mathrm{d}L(\delta\hat{\gamma}(t))\mathrm{d}t,$$

and imposing it to be zero in order for γ to extremize I,

$$\delta I[\gamma] = 0.$$

This problem is easily solved in local coordinates and the result is that $\gamma : [0,1] \to M$ is the solution of the Euler-Lagrange equation corresponding to L. Let $\delta\gamma$ be a $\vartheta-$virtual displacement of γ, and consider the parametrizations $\hat{\gamma}(t) = (\gamma(t), \dot{\gamma}(t)) = (x_1(t),\dots,x_n(t), y_1(t),\dots,y_n(t))$ and $\vartheta(t) = (\vartheta^1(t),\dots,\vartheta^n(t))$, then:

$$\begin{aligned}
0 &= \int_0^1 \mathrm{d}L\big(\delta\hat{\gamma}(t)\big)\,\mathrm{d}t,\\
&= \int_0^1 \left(\frac{\partial L}{\partial x^i}\frac{\partial \vartheta^i}{\partial u} + \frac{\partial L}{\partial y^i}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \vartheta^i}{\partial u}\right)\mathrm{d}t\\
&= \int_0^1 \left(\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^i}\right)\frac{\partial \vartheta^i}{\partial u}\mathrm{d}t + \left.\frac{\partial L}{\partial \dot{\gamma}}\frac{\partial \vartheta^i}{\partial u}\right|_{t=0}^{t=1}.
\end{aligned}$$

Since virtual displacements are null at end points (they are fixed under variations, i.e., $\vartheta(0,u) = \gamma(0)$ and $\vartheta(1,u) = \gamma(1)$), the last term is zero:

$$\int_0^1 \left(\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^i}\right)\frac{\partial \vartheta^i}{\partial u}\mathrm{d}t = 0.$$

At this point we require the *fundamental lemma of variational calculus*. The proof is found in [Lee et al., 2017].

Lemma 2.2.1 (Fundamental lemma of variational calculus)**.** *If a function* $f : \mathbb{R}^n \supset U \to \mathbb{R}$ *is such that*

$$\int_U f(x)g(x)\mathrm{d}x = 0,$$

for all compactly supported smooth function $g : U \to \mathbb{R}$*, then* $f \equiv 0$.

Taking into account arbitrary virtual displacements $\delta\gamma$, by the fundamental lemma of variational calculus, we get the EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0, \qquad i = 1,\dots,\dim M. \tag{2.15}$$

□

Our aim is to get the Euler-Lagrange equation in intrinsic form. The intrinsic version of the EL equation has as solution vector fields on TM whose integral curves project to M to solutions of the coordinate version. Therefore, we require to substitute the constructions of variations and virtual displacements for analogous constructions in the language of vector fields to substitute the derivation in coordinates.

2.2.1 Intrinsic Euler-Lagrange equation

To derive the intrinsic EL equation we need to characterize vector fields on TM, whose flows deform curves $(\gamma, \dot{\gamma}) : [0,1] \to TM$ to variations of these curves. We need to do this in such a way that each deformed curve is an integral curve of a semispray. The following lemma aims to this purpose.

Lemma 2.2.2. *Let $\gamma : [0,1] \to M$ be an integral curve of a semispray $\mathscr{X} \in \mathfrak{X}(TM)$ and consider another vector field $\mathscr{Y} \in \mathfrak{X}(TM)$. Let $\mathrm{Fl}^{\mathscr{X}}$ and $\mathrm{Fl}^{\mathscr{Y}}$ be the flows of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then, the vector field*

$$\mathscr{Z} := \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Fl}^{\mathscr{Y}}_u \circ \mathrm{Fl}^{\mathscr{X}}_t\right), \tag{2.16}$$

for fixed $u \in \mathbb{R}$, defines a semispray if and only if

$$\mathbb{J}[\mathscr{Y}, \mathscr{X}] = 0_{TTM}. \tag{2.17}$$

Proof. Expanding the vector field around $u = 0$ we have,

$$\begin{aligned}
\mathscr{Z} &= \sum_{k=0}^{\infty} \frac{u^k}{k!} \left.\frac{\mathrm{d}^k}{\mathrm{d}u^k}\right|_{u=0} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Fl}^{\mathscr{Y}}_u \circ \mathrm{Fl}^{\mathscr{X}}_t\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Fl}^{\mathscr{Y}}_0 \circ \mathrm{Fl}^{\mathscr{X}}_t\right) + u \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Fl}^{\mathscr{Y}}_u \circ \mathrm{Fl}^{\mathscr{X}}_t\right) + o\left(u^2\right) \\
&= \mathscr{X} + u \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \left(\left(\mathrm{Fl}^{\mathscr{Y}}_{-u}\right)^* \circ \mathscr{X}\right) + o\left(u^2\right) \\
&= \mathscr{X} + u[\mathscr{Y}, \mathscr{X}] + o\left(u^2\right).
\end{aligned}$$

We ignore the terms in $o\left(u^2\right)$. Since $\mathscr{X}$ is a semispray, we have that $\mathbb{J}\mathscr{Z} = \mathscr{V}$ if and only if $\mathbb{J}[\mathscr{Y}, \mathscr{X}] = 0_{TTM}$. □

To complete the characterization of vector field generating variations of $(\gamma, \dot{\gamma})$, we note that such vector fields have to be vertical on the extremes of the curves. Indeed, $\hat{\vartheta}$−virtual displacements $\delta\hat{\gamma}$ are vector fields along the curve $\hat{\gamma} : [0,1] \to TM$ such that $\mathbb{J}\delta\hat{\gamma}(0) = \mathbb{J}\delta\hat{\gamma}(1) = 0_{TTM}$; indeed,

$$\mathbb{J}\delta\hat{\gamma}(t) = \left(\gamma(t), \dot{\gamma}(t); 0_{TM}, \left.\frac{\partial}{\partial u}\right|_{u=0} \vartheta(t,u)\right), \tag{2.18}$$

and the claim follows from the definition of variations of γ (condition b. in definition 2.2.1). Therefore, if $\mathscr{Y} \in \mathfrak{X}(TM)$ is a vector field generating variation of $\hat{\gamma} = (\gamma, \dot{\gamma}) : [0,1] \to TM$, we require them to satisfy

$$\mathbb{J}\mathscr{Y}(\hat{\gamma}(0)) = \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) = 0_{TTM}, \tag{2.19}$$

in addition to 2.17. Due to the fact that vector fields replacing virtual displacements are not arbitrary (they require to satisfy (2.17)) we can not use the fundamental lemma of variational calculus. The following lemma serves for the purpose of allowing the use of the fundamental lemma of variational calculus in the proof of the theorem by means of describing arbitrary vector fields in a convenient manner.

Lemma 2.2.3. *Every vector field $\mathscr{W} \in \mathfrak{X}(TM)$ can be written as $\mathscr{W} = \mathscr{Y} + \mathbb{J}\mathscr{Z}$, where $\mathscr{Y} \in \mathfrak{X}(TM)$ satisfies (2.17) and $\mathscr{Z} \in \mathfrak{X}(TM)$ is an arbitrary vector field.*

Proof. The proof lies on the fact that vector fields $\mathscr{Y} \in \mathfrak{X}(TM)$ satisfying (2.17) are of the form $\mathscr{Y} = (x, y; A, \mathcal{L}_{\mathscr{X}} A)$ where $\mathscr{X}$ is the semispray in (2.17). To see this, write the semispray $\mathscr{X}$ as $\mathscr{X} = (x, y; y, -2G(x,y))$ and a vector field $\mathscr{Y}$ as $\mathscr{Y} = (x, y; A(x,y), B(x,y))$. Then, calculate the bracket,

$$\left[y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, A^j(x,y)\frac{\partial}{\partial x^j} + B^j(x,y)\frac{\partial}{\partial y^j} \right] = \left(y^i \frac{\partial A^j}{\partial x^i} - 2G^i \frac{\partial A^j}{\partial y^i} - B^j \right) \frac{\partial}{\partial x^j} + \left(\cdots \right) \frac{\partial}{\partial y^j}$$

and then,

$$\mathbb{J}[\mathscr{X}, \mathscr{Y}] = \left(y^i \frac{\partial A^j}{\partial x^i} - 2G^i \frac{\partial A^j}{\partial y^i} - B^j \right) \frac{\partial}{\partial y^j}.$$

Hence, in order for $y^i \frac{\partial A^j}{\partial x^i} - 2G^i \frac{\partial A^j}{\partial y^i} - B^j$ to be zero, we have that $B(x,y) = \mathcal{L}_{\mathscr{X}} A$. Therefore, we can write every vector field $\mathscr{W} = (x, y; A(x,y), W(x,y)) \in \mathfrak{X}(TM)$ as $\mathscr{W} = \mathscr{Y} + \mathbb{J}\mathscr{Z}$, where $\mathscr{Y} = (x, y; A(x,y), \mathcal{L}_{\mathscr{X}} A(x,y))$ and $\mathscr{Z}$ is of the form $\mathscr{Z} = (x, y; W(x,y) - \mathcal{L}_{\mathscr{X}} A(x,y), C(x,y))$. □

The main theorem We are ready to enunciate and prove the result that we have announced: the EL equation is the local form the equation $\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L = 0$ (Remark 2.1.1). This is, the integral curves $\hat{\gamma} = (\gamma, \dot{\gamma}) : [0,1] \to TM$ of the vector field $\mathscr{X} \in \mathfrak{X}(TM)$ are the ones that extrimizes the action functional $\int L$. The Remark 2.1.1 gives the result for the particular case of regular lagrangians, i.e., for when the lagrangian 2−form Ω_L is symplectic. Note that the proof of the following theorem does not requires this condition.

Theorem 2.2.1. *Let $L : TM \to \mathbb{R}$ be a lagrangian function and $\Omega_L \in \Omega^1(TM)$ the corresponding lagrangian 2−form. Then, the integral curves $\hat{\gamma} : [0,1] \to TM$ of a semispray $\mathscr{X} \in \mathfrak{X}(TM)$ extremize the funcitonal*

$$\int_{\hat{\gamma}} L \tag{2.20}$$

if and only if

$$\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L = 0. \tag{2.21}$$

Proof. We consider a curve $\hat{\gamma} = (\gamma, \dot{\gamma}) : [0,1] \to M$ and vector field $\mathscr{Y} \in \mathfrak{X}(TM)$ generating variations of this curve, i.e., $\mathscr{Y}$ satisfies

$$\mathbb{J}[\mathscr{Y}, \mathscr{X}] = 0_{TTM}, \quad \text{and} \quad \mathbb{J}\mathscr{Y}(\hat{\gamma}(0)) = \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) = 0_{TTM}$$

after Proposition 2.2.2 and Equation (2.19). We denote by $\hat{\vartheta}(t,u)$ the variations of $\hat{\gamma}(t)$ generated by $\mathscr{Y}$. Then, as in the solution of $\mathscr{P}_1$, we impose the variational derivative to be zero and compute:

$$0 = \left.\frac{\partial}{\partial u}\right|_{u=0} \int_{\hat{\vartheta}(t,u)} L = \int_{\hat{\gamma}} \mathcal{L}_{\mathscr{Y}}(L) = \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} = \left(\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y}\right) - \mathrm{d}L \circ \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) + \mathrm{d}L \circ \mathbb{J}\mathscr{Y}.$$

The added terms in the last equality are zero since $\mathbb{J}\mathscr{Y}(\hat{\gamma}(0)) = \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) = 0$. We put these terms inside the integral by using the fundamental theorem of calculus:

$$\begin{aligned}
\left(\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y}\right) - \mathrm{d}L \circ \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) + \mathrm{d}L \circ \mathbb{J}\mathscr{Y} &= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - \mathrm{d}(\mathrm{d}L \circ \mathbb{J}\mathscr{Y}) \circ \mathscr{X} \\
&= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - \mathcal{L}_{\mathscr{X}}(\mathrm{d}L \circ \mathbb{J}\mathscr{Y}) \\
&= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\mathcal{L}_{\mathscr{X}} \Theta_L) \circ \mathscr{Y} - \Theta_L \circ (\mathcal{L}_{\mathscr{X}} \mathscr{Y}) \\
&= 0.
\end{aligned}$$

The last equality is obtained from $\Theta_L = \mathrm{d}L \circ \mathbb{J}$ and the Leibniz rule for the Lie derivative. Moreover,

$$\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\mathcal{L}_{\mathscr{X}} \Theta_L) \circ \mathscr{Y} - \Theta_L \circ (\mathcal{L}_{\mathscr{X}} \mathscr{Y}) = \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\mathcal{L}_{\mathscr{X}} \Theta_L) \circ \mathscr{Y} - \mathrm{d}L \circ \mathbb{J}[\mathscr{X}, \mathscr{Y}]$$

and since $\mathbb{J}[\mathscr{X}, \mathscr{Y}] = 0_{TTM}$, we get

$$\begin{aligned}
\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\mathcal{L}_{\mathscr{X}} \Theta_L) \circ \mathscr{Y} - \mathrm{d}L \circ \mathbb{J}[\mathscr{X}, \mathscr{Y}] &= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\mathcal{L}_{\mathscr{X}} \Theta_L) \circ \mathscr{Y} \\
&= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\imath_{\mathscr{X}} \mathrm{d} + \mathrm{d}\imath_{\mathscr{X}})\Theta_L \circ \mathscr{Y} \\
&= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\imath_{\mathscr{X}} \Omega_L + \mathrm{d}\imath_{\mathscr{X}}(\mathrm{d}L \circ \mathbb{J})) \circ \mathscr{Y} \\
&= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\imath_{\mathscr{X}} \Omega_L + \mathrm{d}(\mathrm{d}L \circ \mathbb{J}\mathscr{X})) \circ \mathscr{Y}.
\end{aligned}$$

Since $\mathscr{X}$ is a semispray we have $\mathbb{J}\mathscr{X} = \mathscr{V}$ and by using the definition of the energy function $E_L = \mathcal{L}_{\mathscr{V}} L - L$, we get

$$\begin{aligned}
\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\imath_{\mathscr{X}} \Omega_L + \mathrm{d}(\mathrm{d}L \circ \mathbb{J}\mathscr{X})) \circ \mathscr{Y} &= \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} - (\imath_{\mathscr{X}} \Omega_L + \mathrm{d}(\mathrm{d}L \circ \mathscr{V})) \circ \mathscr{Y} \\
&= -\int_{\hat{\gamma}} (\mathrm{d}E_L + \imath_{\mathscr{X}} \Omega_L) \circ \mathscr{Y}.
\end{aligned}$$

The field $\mathscr{Y}$ is not arbitrary, hence we can not claim that $\mathrm{d}E_L + \imath_{\mathscr{X}}\Omega_L$ has to be zero by using the fundamental lemma of variational calculus. However, due to the fact (proved in the following chapter) that the 1−form $\mathrm{d}E_L + \imath_{\mathscr{X}}\Omega_L$ is semibasic (Proposition 3.2.1) and noting that we can generate arbitrary vector fields, due to the Lemma 2.2.3, adding a vertical vector field $\mathbb{J}\mathscr{Z}$ to $\mathscr{Y}$, where $\mathscr{Z} \in \mathfrak{X}(M)$ is arbitrary, we achieve the result. Since $\mathrm{d}E_L + \imath_{\mathscr{X}}\Omega_L$ is a semibasic 1−form, it annihilates $\mathbb{J}\mathscr{Z}$. Therefore,

$$\int_{\hat{\gamma}} (\mathrm{d}E_L + \imath_{\mathscr{X}}\Omega_L) \circ (\mathscr{Y} + \mathbb{J}\mathscr{Z}) = 0$$

for $\mathscr{Y}$ generating variations of $\hat{\gamma}$ and arbitrary $\mathscr{Z} \in \mathfrak{X}(TM)$ if and only if

$$\imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L = 0.$$

□

We finish this chapter by mentioning a result of Chapter 3: *the Lagrange-d'Alembert principle* (Theorem 3.2.2). This result generalizes Theorem 2.2.1 in order to include into the integral a semibasic 1−form $\omega \in \Omega^1(TM)$ after taking the variational derivative of the action functional, i.e.,

$$\delta \int_{\hat{\gamma}} L + \int_{\hat{\gamma}} \omega(\mathscr{Y}) = 0.$$

In this way, we get a generalization of the intrinsic Euler-Lagrange equation,

$$\imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L = \omega,$$

where $\mathscr{X} \in \mathfrak{X}(TM)$ is still a semispray, thus, allowing to model physical systems.

Chapter 3

Lagrange geometry

In this chapter we will develop more properties of the geometric structure introduced in Chapter 2, namely, the (pre)symplectic lagrangian 2−form

$$\Omega_L = \mathrm{d}(\mathcal{L}_{\mathbb{J}} L),$$

associated with a lagrangian function $L: TM \to \mathbb{R}$. All these constructions and properties only depend on the lagrangian function and the configuration manifold M in conjunction with canonical objects of the tangent bundle $TM \to M$. Therefore, it is natural to regard the pair (M, L) as the object of study, in the following called *Lagrange space*.

The section *Lagrange spaces* includes the definition of Lagrange space and some properties that facilitate calculations. In the section *Mechanics* we make sense of the concept *force* as semibasic 1−forms; this drives us to a formulation of the Lagrange-d'Alembert principle which provides a generalization of the intrinsic EL equation, following [Grifone and Mehdi, 1999].

We consider only regular lagrangians so the 2−form Ω_L is symplectic. First, we recall some concepts of Section 2.1.

Remainder of definitions Consider a manifold M and a $C^\infty(TM)$ function $L: TM \to \mathbb{R}$. Then, M is called *configuration manifold*; TM, *phase space* and $L: TM \to \mathbb{R}$, *lagrangian function*. Moreover, let $\mathbb{J}: TTM \to TTM$ be the vertical endomorphism, then we have the following definitions:

- The *lagrangian* 1−*form* and the *lagrangian* 2−*form* are (Definition 2.1.1)

$$\Theta_L := \mathcal{L}_{\mathbb{J}} L \quad \text{and} \quad \Omega_L := \mathrm{d}\Theta_L = \mathrm{d}(\mathcal{L}_{\mathbb{J}} L),$$

  respectively.

- The *energy function associated with* L is (Definition 2.1.3)

$$E_L : TM \to \mathbb{R},\ E_L = \mathcal{L}_{\mathscr{V}} L - L,$$

where $\mathscr{V} \in \mathfrak{X}(TM)$ is the Liouville vector field (Definition 1.2.4).

- A *regular lagrangian* is a lagrangian function $L : TM \to M$ such that the lagrangian 2−form Ω_L is non-degenerate. (Definition 2.1.2)

3.1 Lagrange spaces

The purpose of this section is to define an important canonical vector field to each lagrangian function together with its configuration manifold, the *canonical semispray*. Given a regular lagrangian function $L : TM \to \mathbb{R}$, we know from Corollary 2.1.1 and Remark 2.1.1 that the solution $\mathscr{X} \in \mathfrak{X}(TM)$ to

$$\imath_{\mathscr{X}} \Omega_L = -\mathrm{d}E_L, \tag{3.1}$$

exists, is unique and is a semispray. Recall that integral curves $\hat{\gamma}$ of semisprays project under $\tau_M : TM \to M$ to curves such that they velocity vectors are the same curves $\hat{\gamma}$ on TM (Corollary 1.3.1), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tau_M \circ \hat{\gamma}(t)) = \hat{\gamma}(t), \tag{3.2}$$

which in local coordinates $(x, y) \in TM$, for $\mathscr{X} = (x, y; y, -2G(x, y))$, translates to

$$\frac{\mathrm{d}^2 x^i(t)}{\mathrm{d}t^2} = -2G^i(x, y), \tag{3.3}$$

evoking Newton's second law. Indeed, the integral curves of the semispray $\mathscr{X}$ are of physical interest, since they extremize the functional $\int_{\hat{\gamma}} L$ (Theorem 2.2.1).

These properties are canonical to the given configuration manifold together with the lagrangian function. This motivates the definition of a *Lagrange space* as the pair (M, L), and the definition of *canonical semispray* as the one whose integral curves extremize the functional $\int_{\hat{\gamma}} L$. First, we show some properties of the symplectic Lagrange spaces and then we reserve a subsection *The canonical semispray*.

Definition 3.1.1. A **Lagrange space** is a pair $L^n := (M, L)$ where M is a n−dimensional manifold and $L : TM \to \mathbb{R}$ is a *regular* lagrangian function, i.e, such that the 2−form $\Omega_L = \mathrm{d}\mathcal{L}_{\mathbb{J}}Ł$ is non-degenerate.

The following are properties of the lagrangian 2−form associated with a given Lagrange space useful for calculations.

Proposition 3.1.1. *Consider a Lagrange space* (M, L) *and let* $\Omega_L \in \Omega^2(TM)$ *be the corresponding lagrangian* 2−*form. Then, for any vector fields* $\mathscr{X}, \mathscr{Y} \in \mathfrak{X}(TM)$*, we have*

$$\Omega_L(\mathscr{X}, \mathscr{Y}) = (\mathscr{X} \circ \mathbb{J}\mathscr{Y} - \mathscr{Y} \circ \mathbb{J}\mathscr{X} - \mathbb{J}[\mathscr{X}, \mathscr{Y}])(L). \tag{3.4}$$

Proof. By direct calculation,

$$\begin{aligned}\Omega_L(\mathscr{X}, \mathscr{Y}) &= (\mathrm{d}(\mathrm{d}L \circ \mathbb{J}))(\mathscr{X}, \mathscr{Y}) \\ &= \mathscr{X}(\mathrm{d}L(\mathbb{J}\mathscr{Y})) - \mathscr{Y}(\mathrm{d}L(\mathbb{J}\mathscr{X})) - \mathrm{d}L(\mathbb{J}[\mathscr{X}, \mathscr{Y}]) \\ &= (\mathscr{X} \circ \mathbb{J}\mathscr{Y} - \mathscr{Y} \circ \mathbb{J}\mathscr{X} - \mathbb{J}[\mathscr{X}, \mathscr{Y}])(L).\end{aligned}$$

□

Proposition 3.1.2. *The insertion of the vertical endomorphism annihilates* Ω_L*, i.e.,*

$$\imath_{\mathbb{J}}\Omega_L = 0, \tag{3.5}$$

or equivalently, for any $\mathscr{X}$, $\mathscr{Y} \in \mathfrak{X}(TM)$*:*

$$\Omega_L(\mathbb{J}\mathscr{X}, \mathscr{Y}) + \Omega_L(\mathscr{X}, \mathbb{J}\mathscr{Y}) = 0. \tag{3.6}$$

Proof. By definition of the insertion of vector-valued 1-forms,

$$\imath_{\mathbb{J}}\Omega_L(\mathscr{X}, \mathscr{Y}) = \Omega_L(\mathbb{J}\mathscr{X}, \mathscr{Y}) - \Omega_L(\mathbb{J}\mathscr{Y}, \mathscr{X}).$$

Moreover, by Proposition 3.1.1,

$$\begin{aligned}\Omega_L(\mathbb{J}\mathscr{X}, \mathscr{Y}) + \Omega_L(\mathscr{X}, \mathbb{J}\mathscr{Y}) &= (\mathbb{J}\mathscr{X} \circ \mathbb{J}\mathscr{Y} - \mathscr{Y} \circ \mathbb{J}^2\mathscr{X} - \mathbb{J}[\mathbb{J}\mathscr{X}, \mathscr{Y}] \\ &\quad + \mathscr{X} \circ \mathbb{J}^2\mathscr{Y} - \mathbb{J}\mathscr{Y} \circ \mathbb{J}\mathscr{X}) - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}])(L) \\ &= ([\mathbb{J}\mathscr{X}, \mathbb{J}\mathscr{Y}] - \mathbb{J}[\mathbb{J}\mathscr{X}, \mathscr{Y}] - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}])L \\ &= \mathcal{N}_{\mathbb{J}}(\mathscr{X}, \mathscr{Y})(L) = 0.\end{aligned}$$

□

The following lemma is used in the next subsection to show intrinsically that the hamiltonian vector field associated with the energy function is a semispray.

Lemma 3.1.1. *Consider the lagrangian* 2−*form* Ω_L *and the energy* $E_L = \mathcal{L}_{\mathscr{V}} L - L$*. Then,*

$$\imath_{\mathscr{V}}\Omega_L = \mathcal{L}_{\mathbb{J}} E_L. \tag{3.7}$$

Proof. Since the Lie derivative commutes with the exterior derivative, we have that $[\mathcal{L}_{\mathbb{J}}, \mathrm{d}] = \mathcal{L}_{\mathbb{J}} \circ \mathrm{d} + \mathrm{d} \circ \mathcal{L}_{\mathbb{J}} = 0$. Thus, $\imath_{\mathscr{V}} \Omega_L = \imath_{\mathscr{V}} \mathrm{d}(\mathcal{L}_{\mathbb{J}} L) = -\imath_{\mathscr{V}} \mathcal{L}_{\mathbb{J}} \mathrm{d}L$. Then, the formula (A.5) says that $\imath_{\mathscr{V}} \mathcal{L}_{\mathbb{J}} + \mathcal{L}_{\mathbb{J}} \imath_{\mathscr{V}} = \imath_{\mathbb{J}}$. Therefore,

$$\imath_{\mathscr{V}} \Omega_L = (\mathcal{L}_{\mathbb{J}} \imath_{\mathscr{V}} - \imath_{\mathbb{J}}) \mathrm{d}L = \mathcal{L}_{\mathbb{J}} (\mathcal{L}_{\mathscr{V}}(L) - L) = \mathcal{L}_{\mathbb{J}} E_L.$$

□

3.1.1 The canonical semispray

For each Lagrange space, the hamiltonian vector field associated with the energy function deserves a special name. First it is important to note that this vector field is a semispray, as has been mentioned above. We provide an intrinsic prove of this fact and then we define this vector field as the *canonical semispray*.

Proposition 3.1.3. *Consider a Lagrange space (M, L). The hamiltonian vector field associated with the energy function is a semispray.*

Proof. We need to show the vector field $\mathscr{X}_o \in \mathfrak{X}(TM)$ such that $\imath_{\mathscr{X}_o} \Omega_L = -\mathrm{d}E_L$ satisfies $\mathbb{J}\mathscr{X}_o = \mathscr{V}$. First, we have $\imath_{\mathbb{J}} \imath_{\mathscr{X}_0} \Omega_L = -\imath_{\mathbb{J}} \mathrm{d}E_L = -\mathcal{L}_{\mathbb{J}} E_L$. Now, by the insertion of $\mathbb{J}$ at both sides of $\imath_{\mathscr{X}_o} \Omega_L = -\mathrm{d}E_L$, from formula (A.9) and Proposition 3.1.2 we have that the left side is

$$\imath_{\mathbb{J}} \imath_{\mathscr{X}_o} \Omega_L = (\imath_{\mathscr{X}} \imath_{\mathbb{J}} - \imath_{\mathbb{J}\mathscr{X}}) \Omega_L = -\imath_{\mathbb{J}\mathscr{X}} \Omega_L,$$

and from Lemma 3.1.1 we have that the right side is

$$-\imath_{\mathbb{J}} \mathrm{d}E_L = -\mathcal{L}_{\mathbb{J}} E_L = -\imath_{\mathscr{V}} \Omega_L.$$

Hence, we conclude that $\imath_{\mathbb{J}\mathscr{X}} \Omega_L = \imath_{\mathscr{V}} \Omega_L$ and therefore, $\mathbb{J}\mathscr{X} = \mathscr{V}$. □

Now the definition of *canonical semispray* can be given.

Definition 3.1.2. The **canonical semispray** or **inertial semispray** of a Lagrange space $L^n := (M, L)$ is the semispray $\mathscr{X}_o \in \mathfrak{X}(TM)$ such that:

$$\imath_{\mathscr{X}_o} \Omega_L + \mathrm{d}E_L = 0, \tag{3.8}$$

i.e., the hamiltonian vector field on the symplectic manifold (TM, Ω_L) associated with the hamiltonian function E_L.

The canonical semispray is used, for example, in the next section to construct semisprays relevant for analytical mechanics. The idea is to associate the zero 1−form to the canonical semispray and a vertical vector field to any other semibasic 1−form. In this way, while vertical vector fields are accelerations, we see that 1−forms are a mathematical definition of force; then, the canonical semispray corresponds to the dynamics *free of forces*. We finish the section by proving that if the lagrangian is homogeneous of degree 2, then the canonical semispray is a spray, i.e., an homogeneous vector field of degree 2, i.e., $\mathcal{L}_{\mathscr{V}} \mathscr{X}_o = \mathscr{X}_o$.

Proposition 3.1.4. *If in a given Lagrange space* (M, L) *the lagrangian is homogeneous of degree* 2 *(i.e.,* $\mathcal{L}_{\mathscr{V}} L = 2L$*), then the lagrangian* 2−*form is homogeneous of degree* 2 *and the canonical semispray is a spray; these properties are*

$$\mathcal{L}_{\mathscr{V}} \Omega_L = \Omega_L \qquad \text{and} \qquad \mathcal{L}_{\mathscr{V}} \mathscr{X}_o = [\mathscr{V}, \mathscr{X}_o] = \mathscr{X}_o, \tag{3.9}$$

respectively.

Proof. The proof is found in [De León and Rodrigues, 2011]. □

3.2 Mechanics

In this section we give a mathematical definition of *force* and we relate it to the notion of *acceleration*, as Newton's second law. This allow us to generate new semisprays from a given one, say the canonical semispray. More precisely, we can modify the canonical semispray adding vertical vector fields to it, leading to a new semispray, but the equality $\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_l = 0$ no longer holds. Nevertheless, we see that $\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L$ is semibasic and that there is a bijective correspondence between vertical vector fields and semibasic forms. Before introducing the formal definitions, we make some comments on these ideas. The main result in this section is Theorem 3.2.2 (Lagrange-d'Alembert principle), which generalizes Theorem 2.2.1

3.2.1 Forces

Recall from the coefficient expression of semisprays that its coefficients represent the second time derivative of the configuration coordinate, i.e., if $\mathscr{X} = (x, y; y, -2G(x, y))$, so that $\ddot{x}^i = -2G^i(x, y)$. Adding vertical vector fields to $\mathscr{X}$ modify $\ddot{x}$. Therefore, we see that vertical vector fields are in accordance with the notion of *accelerations*. Since vertical vector fields are in bijection with semibasic 1−forms (Proposition 3.2.3) via the map $\Omega_L^\flat : \mathfrak{X}(TM) \to \Omega^1(TM)$, we interpret semibasic 1−forms as the *forces* producing these accelerations. The analogy with the physical concepts is

$$\Gamma(\mathbb{V}) \xleftrightarrow{\Omega_L^\flat} \Omega^1_{\mathrm{sb}}(TM) \quad : \quad \text{Accelerations} \longleftrightarrow \text{Forces} \tag{3.10}$$

where $\Omega^1_{\rm sb}(TM)$ denotes the space of semibasic 1−forms. We denote by $\mathfrak{X}_{\rm ss}(TM) := \{\mathscr{X} \in \mathfrak{X}(TM) \mid \tau_{M*}\circ \mathscr{X} = \mathbb{I}_{TM}\}$ the set of semisprays.

Definition 3.2.1. Let $L^n := (M, L)$ be a Lagrange space. The semibasic 1− forms $\Omega^1_{\rm sb}(TM)$ are called **forces**. The set (L^n, ω) is called **mechanical system** and the force $-\omega$, **external force** of the mechanical system.

A way of associating semibasic 1−forms to a semispray $\mathscr{X}$ is just by $\imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L$. This 1−form is called *the lagrangian force* of $\mathscr{X}$. Before giving the definition, we prove that this 1−form is semibasic.

Proposition 3.2.1. *Let $L^n = (M, L)$ be a Lagrange space. Then, for any $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$ the 1−form $\Lambda(\mathscr{X}) \in \Omega^1(TM)$ given by*

$$\Lambda(\mathscr{X}) = \imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L,$$

is semibasic, i.e., a force.

Proof. To prove that $\Lambda(\mathscr{X})$ is semibasic is equivalent to show that it annihilates vertical vector fields (Proposition 1.4.1). We evaluate $\mathbb{J}\mathscr{Y}$ in $\Lambda(\mathscr{X})$ for arbitrary $\mathscr{Y} \in \mathfrak{X}(TM)$ and, by using Proposition 3.1.1, we have

$$\begin{aligned}\imath_{\mathbb{J}\mathscr{Y}}\Lambda(\mathscr{X}) &= \imath_{\mathscr{X}}\Omega(\mathbb{J}\mathscr{Y}) + \mathrm{d}E_L(\mathbb{J}\mathscr{Y})\\ &= (\mathscr{X}\circ\mathbb{J}^2\mathscr{Y} - \mathbb{J}\mathscr{Y}\circ\mathbb{J}\mathscr{X} - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}])(L) + \mathbb{J}\mathscr{Y}(E_L).\end{aligned}$$

Now, the definition of the energy function ($E_L = \mathscr{V}(L) - L$) and the fact that $\mathscr{X}$ is a semispray imply that

$$\begin{aligned}\imath_{\mathbb{J}\mathscr{Y}}\Lambda(\mathscr{X}) &= (-\mathbb{J}\mathscr{Y}\circ\mathbb{J}\mathscr{X} - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] + \mathbb{J}\mathscr{Y}\circ\mathscr{V} - \mathbb{J}\mathscr{Y})(L)\\ &= (-\mathbb{J}\mathscr{Y}\circ\mathscr{V} - \mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] + \mathbb{J}\mathscr{Y}\circ\mathscr{V} - \mathbb{J}\mathscr{Y})(L).\end{aligned}$$

Finally, we just need to use the equality $\mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] = -\mathbb{J}\mathscr{Y}$ from Proposition 1.3.3 to conclude the proof,

$$\begin{aligned}\imath_{\mathbb{J}\mathscr{Y}}\Lambda(\mathscr{X}) &= (\mathbb{J}\mathscr{Y} - \mathbb{J}\mathscr{Y})(L)\\ &= 0.\end{aligned}$$

□

Definition 3.2.2. Let $L^n = (M, L)$ be a Lagrange space and consider the map $\Lambda : \mathfrak{X}_{\rm ss}(TM) \to \Omega^1(TM)$ given by

$$\Lambda : \mathscr{X} \mapsto \Lambda(\mathscr{X}) := \imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L.$$

Then, $\Lambda(\mathscr{X})$ is called the **lagrangian force** associated to $\mathscr{X} \in \mathscr{X}_{\rm ss}(TM)$.

Evidently, if $\mathscr{X}_0$ is the canonical semispray, then $\Lambda(\mathscr{X}_0)$ is the semibasic 1−form representing the Euler-Lagrange equation in invariant form, this is, $\Lambda(\mathscr{X}_0) = \imath_{\mathscr{X}_0}\Omega_L + \mathrm{d}E_L = 0$. In fact, the map $\Lambda : \mathfrak{X}_{\rm ss}(TM) \to \Omega^1_{\rm sb}(TM)$ is a bijection.

Proposition 3.2.2. *Let $L^n := (M, L)$ be a Lagrange space and $\omega \in \Omega^1(TM)$ a semibasic $1-$form. Then, the solution $\mathscr{X} \in \mathfrak{X}(TM)$ of the equation*

$$\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L = \omega \tag{3.11}$$

is a unique semispray. In other words, there exists a unique $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$ such that $\Lambda(\mathscr{X}) = \omega$.

Proof. The solution $\mathscr{X} \in \mathfrak{X}(TM)$ to $\imath_{\mathscr{X}} \Omega_L = -\mathrm{d}E + \omega$ exists and is unique since Ω_L is non-degenerate. Note from the formula (A.4) that

$$\imath_{\mathbb{J}} \imath_{\mathscr{X}} \Omega_L = (\imath_{\mathscr{X}} \imath_{\mathbb{J}} - \imath_{\mathbb{J}\mathscr{X}}) \Omega_L = -\imath_{\mathbb{J}\mathscr{X}} \Omega_L,$$

where we have used Proposition 3.1.2. Moreover, by doing an insertion of $\mathbb{J}$ in both sides of (3.11):

$$\imath_{\mathbb{J}} (\imath_{\mathscr{X}} \Omega + \mathrm{d}E_L) = \imath_{\mathbb{J}} \omega.$$

And $\imath_{\mathbb{J}} \omega = 0$ since ω is semibasic we have $\imath_{\mathbb{J}} \imath_{\mathscr{X}} \Omega_L = -\imath_{\mathbb{J}} \mathrm{d}E_L$. Then, using the equality $\imath_{\mathbb{J}} \imath_{\mathscr{X}} \Omega_L = -\imath_{\mathbb{J}\mathscr{X}} \Omega_L$ and from Lemma 3.1.1:

$$\imath_{\mathbb{J}\mathscr{X}} \Omega_L = \imath_{\mathbb{J}} \mathrm{d}E_L = \mathcal{L}_{\mathbb{J}} E_L = \imath_{\mathscr{V}} \Omega_L.$$

Therefore, $\mathbb{J}\mathscr{X} = \mathscr{V}$ since Ω_L is non-degenerate. □

The following corollary says that the force in correspondence with the canonical semispray via the map $\Lambda : \mathfrak{X}_{\text{ss}}(TM) \to \Omega_{\text{sb}}(TM)$ is the zero $1-$form. We can interpret this fact as saying that the canonical semisprays corresponds to the dynamics free of forces.

Corollary 3.2.1. *If $\omega = 0$, then the solution to $\Lambda(\mathscr{X}) = \omega$ is the canonical semispray of the Lagrange space.*

With Proposition 3.2.2 we get the following important consequences that were mentioned at the beginning of the section.

Corollary 3.2.2. *The lagrangian force map $\Lambda : \mathscr{X}_{ss}(TM) \to \Omega^1(TM)$ establishes an isomorphism between semisprays and semibasic $1-$forms.*

Proof. This is due to Proposition 3.2.1 and Proposition 3.2.2, and to the non-degeneracy of the lagrangian $2-$form. □

Corollary 3.2.3 (Bijection between forces and accelerations)**.** *The map $\Omega_L^\flat : \mathfrak{X}(TM) \to \Omega^1(TM)$ restricted to vertical vector fields, i.e.,*

$$\Omega_L^\flat|_{\mathbb{V}} : \Gamma(\mathbb{V}) \to \Omega^1(TM) \; : \mathscr{X} \mapsto \imath_{\mathscr{X}} \Omega_L, \tag{3.12}$$

is an isomorphism between vertical vector fields and semibasic forms.

Proof. Let $\mathscr{X}_0 \in \mathscr{X}_{ss}(TM)$ be the canonical semispray of $L^n = (M, L)$. Then,

$$\Lambda(\mathscr{X}_0) = \imath_{\mathscr{X}_0}\Omega_L + \mathrm{d}E_L = 0.$$

Any vertical vector field $\mathscr{Y} \in \Gamma(\mathbb{V})$ is of the form $\mathscr{Y} = \mathscr{X} - \mathscr{X}_0$ for some semispray $\mathscr{X} \in \mathscr{X}_{ss}(TM)$. Moreover, there exists a unique semibasic form $\omega \in \Omega(TM)$ such that:

$$\Lambda(\mathscr{X}) = \imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L = \omega.$$

Thus,

$$\omega = \Lambda(\mathscr{X}) - \Lambda(\mathscr{X}_0) = \imath_{\mathscr{X}}\Omega_L + \mathrm{d}E_L - \imath_{\mathscr{X}_0}\Omega_L - \mathrm{d}E_L = \imath_{\mathscr{X}-\mathscr{X}_0}\Omega_L = \imath_{\mathscr{Y}}\Omega_L.$$

Therefore, from $\imath_{\mathscr{Y}}\Omega_L = \omega$ the claim holds. □

Now, we have set up the relation between vertical vector fields and semibasics 1−forms. Hence, have the identification:

$$\Gamma(\mathbb{V}) \xleftrightarrow{\Omega_L^\flat} \Omega^1_{\mathrm{sb}}(TM) \quad : \quad \text{Accelerations} \longleftrightarrow \text{Forces.} \tag{3.13}$$

As a final remark, note that every semispray is written as $\mathscr{X} = \mathscr{X}_0 + \mathscr{Y}$, where $\mathscr{X}_0$ is the canonical semispray and $\mathscr{Y} \in \Gamma(\mathbb{V})$ is some vertical vector field. Thus, if $\Lambda(\mathscr{X}) = \omega \in \Omega^1_{\mathrm{sb}}(TM)$, we have:

$$\Lambda(\mathscr{X}) = \Lambda(\mathscr{X}_0) + \Omega_L^\flat(\mathscr{Y}) = \omega.$$

3.2.2 Lagrange-d'Alembert principle

In this subsection we enunciate the main theorem that allow us to introduce nonholonomic constraints in a mechanical system. The fact that semibasic 1−forms ω are in bijection with vertical vector fields throughout the map $\Omega_L^\flat$, provides a way of generate different dynamics on the configuration manifold M since the solution of $\Lambda(\mathscr{X}) = \omega$ is a semispray. This is in accordance with the Lagrange-d'Alembert principle, which we elaborate in this subsection.

To enunciate the principle, recall from Lemma 2.2.2 and Equation 2.19 that variations of curves $\hat{\gamma} := (\gamma, \dot{\gamma}) : [0,1] \to TM$ are generated by vector fields $\mathscr{Y} \in \mathfrak{X}(TM)$ such that

$$\mathbb{J}[\mathscr{X}, \mathscr{Y}] = 0_{TTM} \qquad \text{and} \qquad \mathbb{J}\mathscr{Y}(\hat{\gamma}(0)) = \mathbb{J}\mathscr{Y}(\hat{\gamma}(1)) = 0_{TTM}, \tag{3.14}$$

where $\mathscr{X} \in \mathfrak{X}(TM)$ is the semispray with integral curve $\hat{\gamma}$. Then, as in Theorem 2.2.1, we found sufficient a necessary conditions for $\hat{\gamma} : [0,1] \to TM$ to extremize the action functional $\int_{\hat{\gamma}} L$ by taking the variational derivative,

$$\int_{\hat{\gamma}} \mathcal{L}_{\mathscr{Y}} L = \int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} = 0. \tag{3.15}$$

The Lagrange-d'Alembert principle extends this condition to include semibasics 1−forms in the integrand, in order to obtain *trajectories of the mechanical system.*

Definition 3.2.3. (**The Lagrange-d'Alembert principle**) The **trajectories** $\hat{\gamma} : [0,1] \to TM$ of a mechanical system (L^n, ω) are integral curves of the semispray $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$ such that the sum of the lagrangian force associated with $\mathscr{X}$ and the external force $-\omega$ equals zero, i.e.,

$$\Lambda(\mathscr{X}) - \omega = 0. \tag{3.16}$$

Work done by a force Before continuing with Lagrange-d'Alembert principle, we introduce a common notion in physics: the *work done by a force*. From this definition we arrive at the *work-energy theorem*. The work-energy theorem provides a geometrical picture of the situation when external forces are involved, since we cannot longer expect that trajectories remain in an level set of the energy function.

Definition 3.2.4. Given a force $\omega \in \Omega^1_{sb}(TM)$ and a curve $\hat{\gamma} : [0,1] \to TM$ the **work done** by the force ω through $\hat{\gamma}$ is defined by

$$\int_{\hat{\gamma}} \omega. \tag{3.17}$$

Theorem 3.2.1 (Work-Energy theorem)**.** *The work done by ω through a trajectory $\hat{\gamma} : [0,1] \to TM$ of a mechanical system (L^n, ω) equals the difference of the energy E_L from $\hat{\gamma}(1)$ to $\hat{\gamma}(0)$, i.e.,*

$$\int_{\hat{\gamma}} \omega = E_L(\hat{\gamma}(1)) - E_L(\hat{\gamma}(0)). \tag{3.18}$$

Proof. The calculus is direct:

$$\int_{\hat{\gamma}} \omega = \int_0^1 \hat{\gamma}^* \Lambda(\mathscr{X}) = \int_0^1 \hat{\gamma}^* (\imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L) = \int_0^1 \hat{\gamma}^* \mathrm{d}E_L = \int_0^1 \mathrm{d}\big(E_L \circ \hat{\gamma}\big) = E_L \circ \hat{\gamma}|_0^1.$$

□

The work-energy theorem provides more insight in the non-free of forces dynamics. As we know, the canonical semispray $\mathscr{X}_0$ is a hamiltonian vector field on (TM, Ω_L) with hamiltonian function E_L, thus, the energy E_L is constant along the flow of $\mathscr{X}_0$, i.e., the trajectories of a mechanical system with no external forces remain in a level set of the energy. The work-energy theorem tells us that in the presence of an external force, the trajectory from one point to another is such that the difference of the value of the energy at the points equals the work done by the force.

Lagrange-d'Alembert principle in integral form We can see that, defining the trajectories of a mechanical system is describing the curves satisfying certain functional condition which resembles the condition for extremizing the action functional $\int_{\hat{\gamma}} L$ but including the external force. This is the theorem we enunciate here.

Theorem 3.2.2 (Lagrange-d'Alembert principle, [Marsden and Ratiu, 1995])**.** *Consider a Lagrange space $L^n := (M, L)$ and let (L^n, ω) be a mechanical system. Then, the curve $\hat{\gamma} : [0,1] \to TM$ is a trajectory of the mechanical system, i.e.,*

$$\Lambda(\mathscr{X}) = \imath_{\mathscr{X}} \Omega_L + E_L = \omega, \qquad \text{where } \mathscr{X}(\hat{\gamma}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\gamma}(t), \tag{3.19}$$

if and only if

$$\delta \int_{\hat{\gamma}} L + \int_{\hat{\gamma}} \omega(\delta\hat{\gamma}) = 0. \tag{3.20}$$

Proof. From the integral (3.20) proceed to expand the term $\int \mathrm{d}L \circ \mathscr{Y}$ as in the proof of Theorem 2.2.1, in this way we get that

$$\int_{\hat{\gamma}} \mathrm{d}L \circ \mathscr{Y} = -\int_{\hat{\gamma}} (\imath_{\mathscr{X}} \Omega_L + E_L) \circ \mathscr{Y} = -\int_{\hat{\gamma}} \Lambda(\mathscr{X}) \circ \mathscr{Y}.$$

Then, adding the external force to the integrand,

$$\int_{\hat{\gamma}} (\mathrm{d}L + \omega) \circ \mathscr{Y} = \int_{\hat{\gamma}} (-\Lambda(\mathscr{X}) + \omega) \circ \mathscr{Y}.$$

The arguments to finish te proof are the same as in Theorem 2.2.1, $\Lambda(\mathscr{X})$ and ω are semibasic, their sum is semibasic. Thus,

$$\int_{\hat{\gamma}} (\Lambda(\mathscr{X}) - \omega) \circ (\mathscr{Y} + \mathbb{J}\mathscr{Z}) = 0,$$

for arbitrary $\mathscr{Z} \in \mathfrak{X}(TM)$ and, due to Lemma 2.2.3, every vector field can be written in the form $\mathscr{Y} + \mathbb{J}\mathscr{Z}$, where $\mathscr{Y}$ satisfies (3.14). Therefore, $\Lambda(\mathscr{X}) = \omega$. □

Corollary 3.2.4. *Suppose the external force of the mechanical system is an exact $1-$form, i.e., $\omega = \mathrm{d}F$ for some $F \in C^\infty(TM)$. Then, the Theorem 3.2.2 reduces to the Theorem 2.2.1 for the case where the lagrangian function is $L + F : TM \to \mathbb{R}$.*

Proof. If $\omega = \mathrm{d}F$ then $\int \mathrm{d}(L+F) \circ \mathscr{Y} = 0$ is the extremal condition for the action functional $\int (L+F)$. □

Remark 3.2.1. Curves satisfying the extremal condition for the lagrangian $L + F$ are integral curves of the semispray $\mathscr{X} \in \mathfrak{X}(TM)$ such that $\imath_{\mathscr{X}} \Omega_{(L+F)} + \mathrm{d}E_{(L+F)} = 0$. With a few calculations we return to (3.19), first

$$\imath_{\mathscr{X}} \Omega_{(L+F)} + \mathrm{d}E_{(L+F)} = \imath_{\mathscr{X}} \mathrm{d}(\mathrm{d}(L+F) \circ \mathbb{J}) + \mathrm{d}(\mathcal{L}_{\mathscr{V}}(L+F) - L - F).$$

Since $\mathrm{d}F$ is semibasic, we have that $\mathrm{d}F \circ \mathbb{J} = 0$ and $\mathcal{L}_{\mathscr{V}} F = \mathrm{d}F \circ \mathscr{V} = 0$. Therefore,

$$\begin{aligned} \imath_{\mathscr{X}} \Omega_{(L+F)} + \mathrm{d}E_{(L+F)} &= \imath_{\mathscr{X}} \mathrm{d}(\mathrm{d}L \circ \mathbb{J}) + \mathrm{d}(\mathcal{L}_{\mathscr{V}} L - L - F) \\ &= \imath_{\mathscr{X}} \Omega_L + \mathrm{d}E_L + \mathrm{d}F = 0. \end{aligned}$$

Chapter 4

Connections

When considering a Lagrange space (M, L) we know that its canonical semispray is of variational nature, i.e., the problem of extremize the functional $\int L$ for a given lagrangian is equivalent to the problem of finding the semispray that has zero lagrange force associated. On the other hand, when considering a mechanical system (M, L, ω) we cannot assure that the semispray $\mathscr{X} \in \mathfrak{X}_{\text{ss}}(TM)$ solving for the Lagrange-d'Alembert problem $\Lambda(\mathscr{X}) = \omega$, is of variational nature.

Curves extremizing a functional are usually called *geodesics*. With this in mind, we characterize geodesics to identify semisprays that has a variational character. Geodesics are given in terms of the *covariant derivative* induced by a *connection*. This becomes useful since we also see a method to determine a connection departing from a semispray.

We begin this chapter with a review of the theory of connections.Connections are a way of decompose a bundle into the vertical subbundle and a *horizontal subbundle*, this is the notion of *Ehresmann connections*. On the tangent bundle TM, besides the characterization of connections on vector bundles as almost-product structures, we have a characterization in terms of the vertical endomorphism. The main interest in this is to relate connections with semisprays. A section is devoted to *Lagrangian connections*, here we decompose the bundle TTM into two lagrangian subbundles. We pay special interest in homogeneous lagrangians of degree 2 since their canonical semisprays provide lagrangian connections, then we relate their geodesics to the geodesics of a metric induced by the lagrangian 2−form.

Our main references for this chapter are [Szilasi et al., 2013] and [Miron and Anastasiei, 2012].

4.1 Connections à la Ehresmann

The Ehresmann's definition of connections provides a very visual manner of seeing them, this is, by splitting the total space of the vector bundle $\tau_{TM} : TTM \to TM$ in two components: the vertical subdundle and a *horizontal subbundle*. Hence, it gives us a fine intuition for the developments in theory of connections. We just need to recall the vertical subbundle $\mathbb{V} = \ker(\tau_{M*} : TTM \to TM)$.

Definition 4.1.1. An **(Ehresmann) connection** or **horizontal subbundle** on TM is a subbundle $\mathbb{H} \subset TTM$ of the vector bundle $\tau_{TM} : TTM \to TM$ complementary to the vertical subbundle $\mathbb{V}$, i.e., $\mathbb{H}$ is such that

$$TTM = \mathbb{H} \oplus \mathbb{V},$$

where $\oplus$ is the Whitney sum.

An easy example clarify even further the concept of connection and some of its properties that we prove afterwards. In this view, we return to the simplest case: the circle.

Example 4.1.1. Consider the circle $M = \mathbb{S}^1$ with coordinate $\varphi \in \mathbb{S}^1$ and for its tangent bundle $(\varphi, y) \in TM = \mathbb{S}^1 \times \mathbb{R}$. We know from Proposition 1.2.1 that $\left\{\frac{\partial}{\partial y}\right\}$ is a basis of $\mathbb{V}$. Hence, we identify that $\left\{\frac{\partial}{\partial \varphi}\right\}$ serves as basis for a horizontal subbundle $\mathbb{H} := \text{span}\left\{\frac{\partial}{\partial \varphi}\right\}$. Hence,

$$T T\mathbb{S}^1 = \mathbb{H} \oplus \mathbb{V}, \qquad \text{where} \qquad \mathbb{H} = \text{span}\left\{\frac{\partial}{\partial \varphi}\right\} \quad \text{and} \quad \mathbb{V} = \text{span}\left\{\frac{\partial}{\partial y}\right\}.$$

In fact, we can recognize $\mathbb{H}_{(\varphi,z)}$ as the tangent space $T_\varphi \mathbb{S}^1$ since $\left\{\frac{\partial}{\partial \varphi}\right\}$ spans the tangent space. This property is common to horizontal subbundles (Proposition 4.1.1). Now, note that adding an extra term to $\frac{\partial}{\partial \varphi}$ in the y direction, say $N(\varphi, y)\frac{\partial}{\partial y}$ where $N \in C^\infty(T\mathbb{S}^1)$, still serves as a basis for a complementary subbundle to the vertical subbundle. Therefore,

$$T T\mathbb{S}^1 = \mathbb{H} \oplus \mathbb{V}, \qquad \text{where} \qquad \mathbb{H} := \text{span}\left\{\frac{\partial}{\partial \varphi} - N(\varphi, x)\frac{\partial}{\partial y}\right\}.$$

The minus sign is a convention. More generally, horizontal subbundles can be constructed in this way, providing a set of function $N_i^j \in C^\infty(TM)$, where $i, j = 1, \ldots, n$, to define *locally* a basis

$$\left\{\frac{\partial}{\partial x^1} - N_1^j \frac{\partial}{\partial y^j}, \ldots, \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \ldots, \frac{\partial}{\partial x^n} - N_n^j \frac{\partial}{\partial y^j}\right\}$$

for a horizontal subbundle. The functions N_i^j are called *local coefficients of the connection*, a basis constructed in this way is called *Berwald basis* and the map of a basis for $T_x M$ to the corresponding Berwald basis is called *horizontal lift*.

In the example we have shown that the fibers of horizontal subbundle are isomorphic to the tangent spaces of the circle. The next proposition establishes this fact for the general situation.

Proposition 4.1.1. *Given a connection* $\mathbb{H}$ *on* TM*, the restriction of* $\tau_{M*}: TTM \to TM$ *to* $\mathbb{H}$ *is a morphism of vector bundles and the restriction to fibers are isomorphisms.*

Proof. Just note that $\ker \tau_{M*} = \mathbb{V}$ and $\operatorname{rank} \mathbb{H} = \dim M$. Therefore, the linear map

$$\tau_{M*}|_{\mathbb{H}_v} : \mathbb{H}_v \to T_{\tau_M(v)}M$$

is a linear isomorphism. □

Then, it is possible to *lift* each vector field on M to a unique vector on TM via the map $\tau_{M*}|_{\mathbb{H}} : \mathbb{H} \to TM$, as stated in the following corollary.

Corollary 4.1.1. *Given a connection* $\mathbb{H}$ *on* TM*, for each vector field* $u \in \mathfrak{X}(M)$ *there exists a unique vector field* $\mathscr{X}^H \in \Gamma(\mathbb{H})$ *such that* $\tau_{M*}(\mathscr{X}^H) = u$*.*

Definition 4.1.2. The **horizontal lift** is defined as the map $\mathrm{hl} : \mathfrak{X}(M) \to \Gamma(\mathbb{H}) \ : \ \mathscr{X} \mapsto \mathscr{X}^H$, where $\mathscr{X}^H$ is such that $\tau_{M*}(\mathscr{X}^H) = \mathscr{X}$.

From this we have that $\tau_{M*} \circ \mathrm{hl} = \mathbb{I}_{TM}$ and then, as we see in the following proposition, the horizontal lift is directly related to the vertical lift.

Proposition 4.1.2. *Let* $\mathbb{H}$ *be a connection on* TM*. Then, the horizontal lift and the vertical lift are related by*

$$\mathbb{J} \circ \mathrm{hl} = \mathrm{vl}. \tag{4.1}$$

Proof. Let $u \in \mathfrak{X}(M)$ and calculating $\mathbb{J} \circ \mathrm{hl}(u) = \boldsymbol{i} \circ \boldsymbol{j} \circ \mathrm{hl}(u) = \boldsymbol{i}(\tau_{TM}(\mathrm{hl}(u)), \tau_{M*}(\mathrm{hl}(u))) = \mathbf{i}(\tau_{TM}(u), u)$ □

Since we have decomposed the second tangent bundle as a Whitney sum $TTM = \mathbb{V} \oplus \mathbb{H}$, we can write each vector field $\mathscr{X} \in \mathfrak{X}(TM)$ as $\mathscr{X} = \mathscr{X}^V + \mathscr{X}^H$, where $\mathscr{X}^V \in \Gamma(\mathbb{V})$ and $\mathscr{X}^H \in \Gamma(\mathbb{H})$, i.e.,

$$\mathfrak{X}(TM) = \Gamma(\mathbb{H}) \oplus \Gamma(\mathbb{V}).$$

Sections of $\mathbb{V}$ are already called *vertical vector fields*, then we call **horizontal** vector fields to sections of $\mathbb{H}$. Next, we introduce a way of decomposing vector fields in these two components.

Definition 4.1.3. Given a connection $\mathbb{H}$ we define the two maps $\mathrm{h} : TTM \to \mathbb{H}$ and $\mathrm{v} : TTM \to \mathbb{V}$ by

$$\mathscr{X} = \mathrm{h}(\mathscr{X}) + \mathrm{v}(\mathscr{X}), \qquad \text{for every } \mathscr{X} \in TTM. \tag{4.2}$$

Then, $\mathrm{h} : TTM \to \mathbb{H}$ and $\mathrm{v} : TTM \to \mathbb{V}$ are called **horizontal** and **vertical projection**, respectively.

Remark 4.1.1. The horizontal and vertical projectors satisfy the following properties:

$$\mathrm{h}^2 = \mathrm{h},\ \mathrm{v}^2 = \mathrm{v},\ \mathrm{v} = \mathbb{I}_{\mathfrak{X}(TM)} - \mathrm{h},\ \mathrm{h} \circ \mathrm{v} = \mathrm{v} \circ \mathrm{h} = 0_{TTM}. \tag{4.3}$$

Consequently, we can characterize a connection by a function $\mathrm{h} : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ such that

$$\mathrm{h} \circ \mathrm{h} = \mathrm{h}, \quad \ker h = \Gamma(\mathbb{V}). \tag{4.4}$$

4.2 Ehresmann connections as vector-valued 1−forms

We begin the section with a characterization of connections in terms of a vector-valued 1−form such that it is an *almost-product structure* and has as vertical subbundle the eigenbundle corresponding to the eigenvalue −1. The interest in determining connections trough a vector-valued 1−form is due to the fact that this provides a convenient characterization in terms of the vertical endomorphism for the particular case of the tangent bundle (Proposition 4.2.2).

Proposition 4.2.1. *A connection on TM is equivalent to an almost-product structure $\mathbb{P} \in \Omega(TM, TTM)$ on TM such the eigenbundle corresponding to the eigenvalue −1 is the vertical subbundle, i.e.,*

a. $\mathbb{P} \circ \mathbb{P} = \mathbb{I}_{TTM}$. *(almost-product property)*

b. $\mathrm{E}_{-1}(\mathbb{P}) := \ker(\mathbb{P} + \mathbb{I}_{TTM}) = \mathbb{V}$.

Proof. Consider the vertical projection v of a connection , define $\mathbb{P} := \mathbb{I}_{TTM} - 2\mathrm{v}$. Then, $\mathbb{P}$ is an almost-product structure, i.e., $\mathbb{P}^2 = \mathbb{I}_{TTM}$, and

$$\mathbb{P}(\mathscr{X}) = -\mathscr{X}, \qquad \text{for every } \mathscr{X} \in \Gamma(\mathbb{V}).$$

Conversely, if $\mathbb{P} \in \Omega(TM, TTM)$ satisfies a. and b., define $\mathrm{h} := \frac{1}{2}(\mathbb{I}_{TTM} + \mathbb{P})$. Then, $\mathrm{h} : TTM \to TTM$ satisfies:

$$\mathrm{h} \circ \mathrm{h} = \mathrm{h}. \quad \text{and} \quad \ker \mathrm{h} = \mathbb{V}.$$

□

Remark 4.2.1. From a connection given by an almost product structure $\mathbb{P} \in \Omega^1(TM, TTM)$ we can characterize the horizontal subbundle as the eigenbundle corresponding to the eigenvalue 1. Thus,

$$\mathrm{E}_{-1}(\mathbb{P}) := \ker(\mathbb{P} + \mathbb{I}_{TTM}) = \mathbb{V}, \quad \text{and} \quad \mathrm{E}_{+1}(\mathbb{P}) := \ker(\mathbb{P} - \mathbb{I}_{TTM}) = \mathbb{H}. \tag{4.5}$$

Remark 4.2.2. It is clear, therefore, that given a connection by an almost-product structure $\mathbb{P} \in \Omega^1(TM, TTM)$, the horizontal and vertical projectors are

$$\mathrm{v} = \frac{1}{2}(\mathbb{I}_{TTM} - \mathbb{P}) \qquad \text{and} \qquad \mathrm{h} = \frac{1}{2}(\mathbb{I}_{TTM} + \mathbb{P}), \tag{4.6}$$

respectively.

Note that vertical vector fields $\mathscr{Y} \in \Gamma(\mathbb{V})$ are the ones that $v(\mathscr{Y}) = \mathscr{Y}$. Analogously, we define *horizontal vector fields* in the following way. A vector field $\mathscr{X} \in \mathfrak{X}(TM)$ is called **horizontal** if $h(\mathscr{X}) = \mathscr{X}$. In this sense, sections of $\mathbb{H} \to TM$ are the horizontal vector fields.

4.2.1 In terms of the vertical endomorphism

The existence of the vertical endomorphism provides an additional characterization of Ehresmann connections on the tangent bundle. This fact facilitate the study of connections in our context and allows to construct connections from a given semispray.

Proposition 4.2.2. *A vector-valued* $1-$*form* $\mathbb{P} \in \Omega(TM, TTM)$ *determines a connection if and only if*

$$\mathbb{J} \circ \mathbb{P} = \mathbb{J}, \quad \textit{and} \quad \mathbb{P} \circ \mathbb{J} = -\mathbb{J}. \tag{4.7}$$

Proof. We prove that $\mathbb{P}$ is an almost-product structure such that $\mathrm{E}_{-1}(\mathbb{P}) = \mathbb{V}$. First, to see that $\mathbb{P}$ is an almost-product structure, observe that

$$\mathbb{J} \circ \mathbb{P} = \mathbb{J} \implies \mathbb{J} \circ (\mathbb{P} - \mathbb{I}_{TTM}) = 0_{TTM} \implies \mathrm{im}(\mathbb{P} - \mathbb{I}_{TTM}) \subset \ker \mathbb{J} = \mathbb{V},$$

and

$$\mathbb{P} \circ \mathbb{J} = -\mathbb{J} \implies (\mathbb{P} + \mathbb{I}_{TTM}) \circ \mathbb{J} = 0_{TTM} \implies \mathrm{im}\mathbb{J} = \mathbb{V} \subset \ker(\mathbb{P} + \mathbb{I}_{TTM}).$$

Thus,

$$\mathrm{im}(\mathbb{P} - \mathbb{I}_{TTM}) \subset \ker(\mathbb{P} + \mathbb{I}_{TTM}) \implies (\mathbb{P} + \mathbb{I}_{TTM}) \circ (\mathbb{P} - \mathbb{I}_{TTM}) = 0_{TTM} \implies \mathbb{P} \circ \mathbb{P} = \mathbb{I}_{TTM},$$

therefore, $\mathbb{P}$ is an almost-product structure. It remains to prove $\mathrm{E}_{-1}(\mathbb{P}) = \ker(\mathbb{P} + \mathbb{I}_{TTM}) = \mathbb{V}$, for which we already have $\mathbb{V} \subset \ker(\mathbb{P} + \mathbb{I}_{TTM})$. Let $\mathscr{X} \in \ker(\mathbb{P} + \mathbb{I}_{TTM})$, then, from (4.7),

$$(\mathbb{P} + \mathbb{I}_{TTM})\mathscr{X} = 0_{TTM} \implies \mathbb{J} \circ (\mathbb{P} + \mathbb{I}_{TTM})\mathscr{X} = 0_{TTM} \implies \mathbb{J}\mathscr{X} = 0_{TTM} \implies \mathscr{X} \subset \mathbb{V},$$

Therefore, $\mathbb{V} = \mathrm{E}_{-1}(\mathbb{P})$. Conversely, if $\mathbb{P}$ is an almost-product structure such that $\ker(\mathbb{P} + \mathbb{I}_{TTM}) = \mathbb{V}$, then we have that vectors $\mathscr{X} \in TTM$ satisfying $(\mathbb{P} + \mathbb{I}_{TTM})\mathscr{X} = 0_{TTM}$ are of the form $\mathscr{X} = \mathbb{J}\mathscr{Y}$. Hence, $\mathbb{P} \circ \mathbb{J} = -\mathbb{J}$. Moreover, since $(\mathbb{P} + \mathbb{I}_{TTM}) \circ (\mathbb{P} - \mathbb{I}_{TTM}) = 0$ we have that vectors in the image of $(\mathbb{P} - \mathbb{I}_{TTM})$ are of the form $\mathbb{J}\mathscr{Y}$. Therefore, $\mathbb{J} \circ (\mathbb{P} - \mathbb{I}_{TTM})$ annihilates every vector in TTM and this finishes the proof. □

4.3 Berwald basis of a connection

Before passing to relate connections to semisprays we introduce an appropriate kind of bases of TTM for doing local calculations when dealing with Ehresmann connections on TM: the *Berwald bases*. An example of these was already elucidated in the example of the circle (Example 4.1.1). We see in the next section, that the coefficients of a semispray provides a simple way of construct the connection determined by the semispray in terms of a Berwald basis (Proposition 4.17).

Proposition 4.3.1. *Let $\mathbb{H}$ be an Ehresmann connection on TM and let $\left\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right\}$ be a basis of T_xM. Then, the set*

$$\left\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n}\right\}, \quad \textit{where } \frac{\delta}{\delta x^i} := \mathrm{hl}\left(\frac{\partial}{\partial x^i}\right),$$

is a basis of $\mathbb{H}_u$, where $\tau_M(u) = x$.

Definition 4.3.1. The basis set defined in Proposition 4.3.1 is called a **Berwald basis** of the connection.

If (x, y) are local coordinates of TM, then a basis for TT_xM is given by $\left\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right\}$, since $\left\{\frac{\partial}{\partial y^i}\right\}$ is a basis for $\mathbb{V}_u$. This is called a **local basis adapted** to the connection.

Remark 4.3.1 (Local coefficients of the connection)*.* Recall that the horizontal lift is an isomorphism from T_xM to H_u, ($\tau_M(u) = x$) and that it gives the unique $\mathscr{X}^H \in \Gamma(\mathbb{H})$ such that $\tau_{M*}(\mathscr{X}^H) = \mathscr{X}$. Therefore, from the condition:

$$\tau_{M*}\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i},$$

we have that the basic elements of $\mathbb{H}$ are locally written as

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}. \tag{4.8}$$

The set of functions $\{N_j^i\}$ are called **local coefficients of the connection**. From this observation, we see inmediatly that the vertical endomorphism acts on Berwald bases as

$$\mathbb{J}\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}. \tag{4.9}$$

The dual basis of the adapted basis is given by $\{\mathrm{d}x, \delta y\}$, where

$$\delta y^i := \mathrm{d}y^i + N_j^i(x, y)\mathrm{d}x^j. \tag{4.10}$$

Now, we show how can we write the local expression for an almost-product structure describing a connection in terms of an adapted local basis.

Proposition 4.3.2. *Let $\mathbb{P}$ be an almost-product structure describing an Ehresmann connection on TM. Then with respect to a local basis $\left\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right\}$ adapted to the connection, we have*

$$\mathbb{P} = \frac{\delta}{\delta x^i} \otimes \mathrm{d}x^i - \delta y^i \otimes \frac{\partial}{\partial y^i}. \tag{4.11}$$

Proof. From the properties in Proposition 4.2.2, the action of $\mathbb{P}$ is

$$\mathbb{P}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \quad \text{and} \quad \mathbb{P}\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial y^i}.$$

This allows to write $\mathbb{P}$ as Equation (4.11). □

With this, we can express the horizontal and vertical projectors in terms of the adapted basis.

Corollary 4.3.1. *The horizontal and vertical projectors of a connection can be written with respect to an adapted basis as*

$$h = \mathrm{d}x^i \otimes \frac{\delta}{\delta x^i}, \qquad \textit{and} \qquad v = \delta y^i \otimes \frac{\partial}{\partial y^i}. \tag{4.12}$$

Proof. Together with te proof of the previous proposition, recall that $\mathbb{V}$ is the eigenbundle corresponding to the eigenvalue -1 of $\mathbb{P}$. □

4.4 Semisprays and Ehresmann connections

Relating semisprays and connections is necessary in order to determine geodesic solutions in mechanical systems, due to the fact that trajectories are integral curves of semisprays. This section has two purposes concerning semisprays and Ehresmann connections: (1) to associate a special semispray to each connection and (2) to determine a connection via a given semispray. At the end, we see the case when these relations are equivalent: if and only if the given semispray is a spray (a semispray homogeneous of degree 2). In this section we see that connections determine a special semispray, called the *semispray associated with the connnection.* We begin associating a semispray with a connection and then determining a connection given a semispray.

4.4.1 Semispray associated with a connection

To make sense of the following definition, first note that, given a semispray $\mathscr{X}$ and a horizontal projector $\mathrm{h} : TTM \to \mathbb{H}$, the vector field $\mathrm{h}(\mathscr{X})$ is a semispray. Indeed, let $\mathbb{P} \in \Omega^1(TM, TTM)$ be the almost-product structure associated with the connection, then we can write $\mathrm{h} = \frac{1}{2}(\mathbb{I}_{TTM} + \mathbb{P})$. Thus,

$$\mathbb{J} \circ \mathrm{h}(\mathscr{X}) = \frac{1}{2}(\mathbb{J} + \mathbb{J} \circ \mathbb{P})(\mathscr{X}) = \mathbb{J}\mathscr{X} = \mathscr{V},$$

taking into consideration Proposition 4.2.2, from where we know $\mathbb{J} \circ \mathbb{P} = \mathbb{J}$. Now, we define the semispray $\mathrm{h}(\mathscr{X})$ as the associated semispray with the connection and then, we show that it is well defined.

Definition 4.4.1. Consider a connection $\mathbb{H}$ and let $\mathrm{h} : TTM \to TTM$ be the associated horizontal projector. Then, the **semispray associated with the connection** is $\mathscr{X}_{\mathbb{H}} \in \mathfrak{X}(TM)$ given by

$$\mathscr{X}_{\mathbb{H}} := \mathrm{h}(\mathscr{X}), \tag{4.13}$$

where $\mathscr{X} \in \mathfrak{X}(TM)$ is an arbitrary semispray.

To prove that the semispray $\mathscr{X}_{\mathbb{H}}$ associated with the connection is well defined, observe that given two semisprays $\mathscr{X}_1$, $\mathscr{X}_2 \in \mathfrak{X}(TM)$, the vector field $\mathscr{X}_1 - \mathscr{X}_2$ is a vertical. Then,

$$\mathrm{h}(\mathscr{X}_1 - \mathscr{X}_2) = 0_{TTM}.$$

Note also that, if $N^i_j(x,y) \in C^\infty(TM)$ are local coefficients of the connection, then the associated semispray is locally

$$\mathscr{X}_{\mathbb{H}} = y^i \frac{\partial}{\partial x^i} - y^j N^i_j(x,y) \frac{\partial}{\partial y^i}. \tag{4.14}$$

4.4.2 Connection determined by a semispray

We proceed to define an Ehresmann connection from a given semispray. For this, we take advantage of the vertical endomorphism $\mathbb{J} : TTM \to TTM$ and the characterization of connections in terms of $\mathbb{J}$. The following proposition introduces already the definition of the connection determined by a semispray.

Proposition 4.4.1. *Consider a semispray $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$. Then, the vector-valued $1-$form*

$$\mathbb{P} := [\mathbb{J}, \mathscr{X}]_{FN} \tag{4.15}$$

determines an Ehresmann connection.

Proof. We show that the vector-valued $1-$form $\mathbb{P} = [\mathscr{X}, \mathbb{J}]_{FN}$ satisfies Proposition 4.2.2, i.e.,

$$\mathbb{J} \circ \mathbb{P} = \mathbb{J}, \qquad \mathbb{P} \circ \mathbb{J} = -\mathbb{J}.$$

From the Frölicher-Nijenhuis calculus (formula A.2.1) we have that

$$\mathbb{P}(\mathscr{Y}) = [\mathbb{J}, \mathscr{X}]_{FN}(\mathscr{Y}) = [\mathbb{J}\mathscr{Y}, \mathscr{X}] - \mathbb{J}[\mathscr{Y}, \mathscr{X}],$$

for any $\mathscr{Y} \in \mathfrak{X}(TM)$. From Proposition 1.3.3 we now that $\mathbb{J}[\mathscr{X}, \mathbb{J}\mathscr{Y}] = -\mathbb{J}\mathscr{Y}$. Therefore, we get

$$\mathbb{J} \circ \mathbb{P}(\mathscr{Y}) = \mathbb{J}[\mathbb{J}\mathscr{Y}, \mathscr{X}] = \mathbb{J}\mathscr{Y} \qquad \text{and} \qquad \mathbb{P} \circ \mathbb{J} = -\mathbb{J}[\mathbb{J}\mathscr{Y}, \mathscr{X}] = -\mathbb{J}\mathscr{Y}.$$

□

Definition 4.4.2. Let $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$ be a semispray. The **Ehresmann connection determined by** $\mathscr{X}$ is $\mathbb{P} := [\mathscr{X}, \mathbb{J}]_{FN}$.

Local coefficients of the connection In view of Proposition 1.3.1, we can write the semispray locally as $\mathscr{X} = (x, y; y, -2G(x, y))$. The local coefficients of the connection determined by $\mathscr{X}$ has the simple expression

$$N^i_j(x, y) = \frac{\partial G^i(x, y)}{\partial y^j}. \tag{4.16}$$

To see this, compute $\mathbb{P}(\mathscr{Y})$ for a vector field $\mathscr{Y} = (x, y; A(x, y), B(x, y))$. We get that

$$\mathbb{P}(\mathscr{Y}) = A^i(x, y)\frac{\partial}{\partial x^i} - \left(B^i(x, y) + 2A^j(x, y)\frac{\partial G^i(x, y)}{\partial y^j}\right)\frac{\partial}{\partial y^i}.$$

Then, just compare the result with the expression for $\mathbb{P}$ in terms of an adapted basis to the connection (Equation 4.11),

$$\mathbb{P} = \mathrm{d}x^i \otimes \frac{\delta}{\delta x^i} - \delta y^i \otimes \frac{\partial}{\partial y^i},$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y)\frac{\partial}{\partial y^j} \qquad \text{and} \qquad \delta y^i = \mathrm{d}y^i + N^i_j(x, y)\mathrm{d}x^j. \tag{4.17}$$

In general, the semispray associated with the connection $[\mathbb{J}, \mathscr{X}]_{FN}$ is not the semispray $\mathscr{X}$. To finish this section, we show that this is the case if and only if $\mathscr{X}$ is a spray, i.e., a homogeneous semispray of degree 2, i.e., $\mathcal{L}_{\mathscr{V}}\mathscr{X} = \mathscr{X}$.

Proposition 4.4.2. *Let a semispray $\mathscr{X} \in \mathfrak{X}(TM)$ and $\mathbb{P} = [\mathbb{J}, \mathscr{X}]_{FN}$ the connection determined by $\mathscr{X}$. Then, the semispray associated with the connection is the same $\mathscr{X}$ if and only if $\mathscr{X}$ is a spray.*

Proof. Choose the semispray $\mathscr{X} \in \mathfrak{X}(TM)$ to compute the horizontal projection,

$$\mathrm{h}(\mathscr{X}) = \frac{1}{2}(\mathscr{X} + [\mathbb{J}, \mathscr{X}]_{FN}(\mathscr{X})) = \frac{1}{2}(\mathscr{X} - [\mathscr{X}, \mathscr{V}]).$$

Then $\mathrm{h}(\mathscr{X}) = \mathscr{X}$ if and only if $\mathcal{L}_{\mathscr{V}}\mathscr{X} = [\mathscr{V}, \mathscr{X}] = \mathscr{X}$. □

4.5 Lagrangian connection

Now we introduce Lagrange spaces and the corresponding lagrangian 2−forms into the study of connection. The natural question is what structure has a horizontal subbundle with respect to the symplectic form $\Omega_L \in \Omega^2(TM)$, given a lagrangian $L : TM \to \mathbb{R}$. Since the rank of horizontal distributions is the dimension of the base manifold M we can only hope that it is lagrangian with respect to Ω_L. If this is the case, then we have a decomposition of the second tangent TTM in lagrangian subbundles since the vertical subbdunle is always lagrangian with respect to lagrangian 2−forms. Indeed, we can see that Ω_L restricted to vertical vector fields is zero,

$$\Omega_L(\mathscr{X},\mathscr{Y}) = (\mathscr{X}\circ\mathbb{J} - \mathscr{Y}\circ\mathbb{J}\mathscr{X} - \mathbb{J}[\mathscr{X},\mathscr{Y}])(L) = 0, \qquad \text{for every } \mathscr{X},\mathscr{Y}\in\Gamma(\mathbb{V}).$$

Connections whose corresponding horizontal subbundle is lagrangian receive the special name of *lagrangian connections*. We study in this section some properties of such connections.

Definition 4.5.1. Let $L^n := (M, L)$ be a Lagrange space. A **lagrangian connection** is a connection $\mathbb{H}\subset TTM$ such that it is a lagrangian subbundle with respect to the lagrangian 2−form Ω_L, i.e.,

$$\Omega_L(\mathrm{h}(\mathscr{X}),\mathrm{h}(\mathscr{Y})) = 0, \tag{4.18}$$

for every $\mathscr{X}$, $\mathscr{Y}\in\mathfrak{X}(TM)$.

The following characterizations of lagrangian connections are useful for calculations.

Proposition 4.5.1. *Let $\mathbb{P}\in\Omega^1(TM, TTM)$ be the almost-product structure determining a connection and $\mathrm{h}: TTM \to TTM$ the horizontal projector. Then, the following statements are equivalent:*

a. The connection is lagrangian.

b. $\imath_{\mathbb{P}}\Omega_L = 0$.

c. $\imath_{\mathrm{h}}\Omega_L = \imath_{\mathrm{v}}\Omega_L = \Omega_L$.

Proof. First, we prove that a. is equivalent to c.

1. We show by direct calcution of $\imath_{\mathrm{h}}\Omega_L$ on vector fields $\mathscr{X},\mathscr{Y}\in\mathfrak{X}(TM)$ that it is equal to $\Omega_L(\mathscr{X},\mathscr{Y})$ if and only if the connection is lagrangian, this is,

$$\begin{aligned}\imath_{\mathrm{h}}\Omega_L(\mathscr{X},\mathscr{Y}) &= \Omega_L(\mathrm{h}(\mathscr{X}),\mathscr{Y}) + \Omega_L(\mathscr{X},\mathrm{h}(\mathscr{Y}))\\ &= \Omega_L(\mathrm{h}(\mathscr{X}),\mathrm{h}(\mathscr{Y})+\mathrm{v}(\mathscr{Y})) + \Omega_L(\mathrm{h}(\mathscr{X})+\mathrm{v}(\mathscr{X}),\mathrm{h}(\mathscr{Y}))\\ &= \Omega_L(\mathrm{h}(\mathscr{X}),\mathrm{v}(\mathscr{Y})) + \Omega_L(\mathrm{v}(\mathscr{X}),\mathrm{h}(\mathscr{Y})) + \Omega_L(\mathrm{h}(\mathscr{X}),\mathrm{h}(\mathscr{Y})) + \Omega_L(\mathrm{v}(\mathscr{X}),\mathrm{v}(\mathscr{Y}))\\ &= \Omega_L(\mathrm{h}(\mathscr{X})+\mathrm{v}(\mathscr{X}),\mathrm{h}(\mathscr{Y})+\mathrm{v}(\mathscr{Y}))\\ &= \Omega_L(\mathscr{X},\mathscr{Y}).\end{aligned}$$

   The calculation for $\imath_{\mathrm{v}}\Omega_L$ is analogous.

2. To prove that b. is equivalent to c. use c. and Proposition 4.3.2. We have that

$$\imath_{\mathbb{P}}\Omega_L = \imath_{\mathrm{h}-\mathrm{v}}\Omega_L = \imath_{\mathrm{h}}\Omega_L - \imath_{\mathrm{v}}\Omega_L = 0.$$

□

In turns out that if we have a lagrangian connection, the symplectic lagrangian 2−form has a simpler expression in terms of an adapted basis to any connection, since evaluating the symplectic form on the adapted basis, we have

$$\Omega_L\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=b_{ij},\qquad \Omega_L\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)=-g_{ij} \tag{4.19}$$

$$\Omega_L\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=g_{ij},\qquad \Omega_L\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=c_{ij}=0, \tag{4.20}$$

and the connection is lagrangian if and only if $b_{ij}=0$. Then, if the connection is lagrangian, we have that

$$\Omega_L=g_{ij}\delta y^i\wedge \mathrm{d}x^j,\qquad \text{where}\quad g_{ij}:=\frac{\partial^2 L}{\partial y^i\partial y^j}, \tag{4.21}$$

which can be seen from the local expression of the lagrangian 2−form,

$$\Omega_L=\frac{\partial^2 L}{\partial x^i\partial y^j}\mathrm{d}x^i\wedge\mathrm{d}x^j+\frac{\partial^2 L}{\partial y^i\partial y^j}\mathrm{d}y^i\wedge\mathrm{d}x^j,$$

by replacing $\mathrm{d}y^i=\delta y^i-N^i_k\mathrm{d}x^k$ into Ω_L. Lagrangian connections indeed always exists, since we already have the lagrangian subbundle $\mathbb{V}$ and we can find a lagrangian complement to this subbundle. Moreover, we can see that in the case of an homogeneous lagrangian $L:TM\to\mathbb{R}$ of degree 2, the canonical semispray of the corresponding Lagrange space determines a lagrangian connection.

Proposition 4.5.2. *Let (M,L) be a Lagrange space where $L:TM\to\mathbb{R}$ is homogeneous of degree 2. Then, the canonical semispray $\mathscr{X}_o$ determines a lagrangian connection, $\mathbb{P}=[\mathbb{J},\mathscr{X}_o]_{FN}$.*

Proof. By using formula (A.10), we have that

$$\imath_{\mathbb{P}}\Omega_L=\imath_{[K,\mathscr{X}]_{\mathrm{FN}}}\Omega_L=(\imath_{\mathscr{X}}\circ\mathcal{L}_{\mathbb{J}}+\mathcal{L}_{\mathbb{J}}\circ\imath_{\mathscr{X}}-\mathcal{L}_{\mathbb{J}\mathscr{X}})\Omega_L=(\mathcal{L}_{\mathbb{J}}\circ\imath_{\mathscr{X}}-\mathcal{L}_{\mathscr{V}})\Omega_L=(-\mathcal{L}_{\mathbb{J}}\mathrm{d}E_L-\mathrm{d}\imath_{\mathscr{V}})\Omega_L.$$

Since $-\mathcal{L}_{\mathbb{J}}\mathrm{d}E_L=\mathrm{d}\mathcal{L}_{\mathbb{J}}E_L$ and $\mathcal{L}_{\mathbb{J}}\mathrm{d}E_L=\imath_{\mathscr{V}}\Omega_L$ by Lemma 3.1.1, we conclude that $\imath_{\mathbb{P}}\Omega_L=0$. □

Recall from Definition 4.4.1 that the semispray associated with the connection $\mathbb{H}$ is given by $\mathscr{X}_{\mathbb{H}}=\mathrm{h}(\mathscr{X})$, where $\mathrm{h}:TTM\to\mathbb{H}$ is the horizontal projector an $\mathscr{X}$ an arbitrary spray. Then, the following proposition states that the semispray $\mathscr{X}_{\mathbb{H}}$ is the canonical semispray if and only if the horizontal subbundle is tangent to the level sets of the energy E_L.

Proposition 4.5.3. *Consider a Lagrange space $L^n:=(M,L)$ and $\mathrm{h}:TTM\to TTM$ the horizontal projector of a lagrangian connection. Then the semispray associated with the connection is the canonical semispray of L^n if and only if:*

$$\mathcal{L}_{\mathrm{h}}E_L=\mathrm{d}E_L\circ\mathrm{h}=0. \tag{4.22}$$

Proof. Let $\mathscr{X}_{\mathbb{H}} \in \mathfrak{X}(TM)$ be the semispray associated with the connection , i.e., $\mathscr{X}_{\mathbb{H}} = \mathrm{h}(\mathscr{X})$ for an arbitrary semispray $\mathscr{X} \in \mathfrak{X}(TM)$. Then, computing:

$$\begin{aligned}
\imath_{\mathrm{h}}(\imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L) &= \imath_{\mathscr{X}_{\mathbb{H}}}\imath_{\mathrm{h}}\Omega_L - \imath_{\mathrm{h}\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L \circ \mathrm{h} \\
&= \imath_{\mathscr{X}_{\mathbb{H}}}\imath_{\mathrm{h}}\Omega_L - \imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L \circ \mathrm{h} && (\mathrm{h}(\mathscr{X}_{\mathbb{H}}) = \mathscr{X}_{\mathbb{H}}) \\
&= \imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L - \imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L \circ \mathrm{h} && (\imath_{\mathrm{h}}\Omega_L = \Omega_L, \text{ proposition 4.5.1 }) \\
&= \mathrm{d}E_L \circ \mathrm{h}.
\end{aligned}$$

Since, $\imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L$ is semibasic we have that $\imath_{\mathrm{h}}(\imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L) = \imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L$, then:

$$\imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + \mathrm{d}E_L = \mathrm{d}E_L \circ \mathrm{h}.$$

Therefore, $\mathscr{X}_{\mathbb{H}}$ is the canonical semispray of L^n, meaning:

$$\imath_{\mathscr{X}_{\mathbb{H}}}\Omega_L + E_L = 0,$$

if and only if $\mathrm{d}E_L \circ \mathrm{h} = 0$. □

4.5.1 The adapted metric

In this subsection we provide a metric tensor on TM such that the horizontal and vertical subbundles are orthogonal with respect to the given metric. In order to define a metric from a given lagrangian 2−form we need to consider the *almost-product structure* associated with a connection. Details of this structure can be consulted on [Antonelli, 2003]. Consider a connection on TM and an adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$, let $\mathbb{F} : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$ be the $C^\infty(TM)$−linear mapping given by

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i} \quad \text{and} \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}. \tag{4.23}$$

This function satisfies the following properties:

1. In terms of the adapted basis it is given by

$$\mathbb{F} = -\mathrm{d}x^i \otimes \frac{\partial}{\partial y^i} + \frac{\delta}{\delta x^i} \otimes \delta y^i. \tag{4.24}$$

2. It is an almost-product structure, i.e., $\mathbb{F}^2 = -\mathbb{I}_{TTM}$.

3. With the horizontal projector $\mathrm{h} : TTM \to \mathbb{H}$, we have

$$\mathbb{F} \circ \mathrm{h} = \mathbb{J} \quad \text{and} \quad \mathbb{F} \circ \mathbb{J} = -\mathrm{h}. \tag{4.25}$$

4. Moreover,

$$\mathbb{F}\circ\mathbb{J}+\mathbb{J}\circ\mathbb{F}=\mathbb{I}_{TTM}. \tag{4.26}$$

The function $\mathbb{F}:\mathfrak{X}(TM)\to\mathfrak{X}(TM)$ is called the **almost-complex structure** associated with the connection. This function is used to define a metric on TM in the following way.

Definition 4.5.2. Let (M,L) be a Lagrange space and $\mathbb{F}:\mathfrak{X}(TM)\to\mathfrak{X}(TM)$ the almost-product structure associated with a connection $\mathbb{P}$ on TM. The **adapted metric** to the connection is the 2−form $g_{\mathbb{P}}\in\Omega^2(TM)$ given by

$$g_{\mathbb{P}}(\mathscr{X},\mathscr{Y}):=\Omega_L(\mathscr{X},\mathbb{F}\mathscr{Y}), \tag{4.27}$$

for every $\mathscr{X},\mathscr{Y}\in\mathfrak{X}(TM)$.

Orthogonality of $\mathbb{H}$ with $\mathbb{V}$. We can readily see that the horizontal and vertical subbundles are orthogonal with respect to $g_{\mathbb{P}}$, i.e., $g_{\mathbb{P}}\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=0$ for every $i,j=1\dots,n$, since the vertical subbundle is lagrangian. Moreover, $g_{\mathbb{P}}$ is non-degenerate due to the non-degeneracy of Ω_L and the definition of $\mathbb{F}$, and is symmetric since $g_{\mathbb{P}}\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)+g_{\mathbb{P}}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=\Omega_L\left(\frac{\delta}{\delta x^i},-\frac{\partial}{\partial y^j}\right)+\Omega_L\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=\Omega_L\left(\frac{\partial}{\partial y^j},\frac{\delta}{\delta x^i}\right)+\Omega_L\left(\frac{\delta}{\delta x^j},-\frac{\partial}{\partial y^i}\right)=g_{\mathbb{P}}\left(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^i}\right)+g_{\mathbb{P}}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)$.

Furthermore, it is immediate to see (by evaluating on an adapted basis) that the coefficients of the metric tensor are precisely the coefficients of the 2−form (4.21). Therefore, in terms of the dual of an adapted basis we can write the adapted metric as

$$g_{\mathbb{P}}=g_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j+g_{ij}\delta y^i\otimes\delta y^j,\qquad\text{where }\ g_{ij}=\frac{\partial L}{\partial y^i\partial y^j}. \tag{4.28}$$

Geodesics of a metric From [Chern et al., 1999], we take the definition of *geodesics of the metric* $g_{\mathbb{P}}$ as the solutions of the equation

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+N^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=0, \tag{4.29}$$

where the functions N^i_{jk} are the so-called *Christoffel symbols* (of the second kind),

$$N^i_{jk}:=\frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k}+\frac{\partial g_{kl}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^l}\right),\qquad\text{and }\ (g^{ij})=(g_{ij})^{-1}. \tag{4.30}$$

4.5.2 Covariant derivative

This section presents the definition of *covariant derivative* and of *geodesics* of a connection. Geodesics are curves in M whose accelerations remain horizontal with respect to a connection . In particular, we see that for a homogeneous lagrangian $L : TM \to \mathbb{R}$ of degree 2 the solutions of the Euler-Lagrange equation are geodesics with respect to the connection determined by the canonical semispray (Proposition 4.5.5).

Definition 4.5.3 ([Grifone and Mehdi, 1999])**.** Let N and M be manifolds and $\mathbb{H}$ a connection on TM. The **covariant derivitarive** ∇ induced by the connection is given by

$$\nabla_{\mathscr{Y}} \mathscr{X} := \mathrm{v} \circ \mathscr{X}_* \circ \mathscr{Y}, \tag{4.31}$$

where $\mathrm{v} : TTM \to \mathbb{V}$ is the vertical projector of the connection, $\mathscr{Y} \in \mathfrak{X}(N)$ and $\mathscr{X}$ a map $\mathscr{X} : N \to TM$.

Example 4.5.1 (Covariant derivative along a curve γ)**.** Let $N = [0,1]$ and $\mathscr{Y} = \frac{\partial}{\partial t}$. Consider a curve $\gamma : [0,1] \to M$ and a vector field $\mathscr{X} : [0,1] \to TM$ along the curve, i.e., $\tau_M \circ \mathscr{X} = \gamma$. Parametrize the curve as $\gamma(t) = (x^1(t), \dots, x^n(t)) \in M$, and the vector field as $\mathscr{X}(t) = (x^1(t), \dots, x^n(t), y^1(t), \dots, y^n(t)) \in T_{\gamma(t)}M$. Then,

$$(\mathscr{X}_*)_t \circ \frac{\partial}{\partial t} = (x(t), y(t); \dot{x}(t), \dot{y}(t)),$$

and

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} \mathscr{X} &= \mathrm{v}(x(t), y(t); \dot{x}(t), \dot{y}(t)) \\
&= \left(\delta y^i \otimes \frac{\partial}{\partial y^i}\right)(x(t), y(t); \dot{x}(t), \dot{y}(t)) \\
&= \left(\mathrm{d}y^i + N^i_j \mathrm{d}x^j\right) \otimes \frac{\partial}{\partial y^i}(x(t), y(t); \dot{x}(t), \dot{y}(t)) \\
&= \left(\dot{y}^i(t) + \dot{x}^j(t) N^i_j(x, y)\right) \frac{\partial}{\partial y^i}.
\end{aligned}$$

Definition 4.5.4. Consider a connection on TM and the induced covariant derivative ∇. A vector field $\mathscr{X} \in \mathfrak{X}(M)$ along a curve is called **parallel** if its covariant derivative is zero, i.e.,

$$\nabla_{\frac{\partial}{\partial t}}(\mathscr{X}) = 0_{TTM}.$$

This notion allows us to introduce a definition for when the accelerations of a given curve $\gamma : [0,1] \to M$ are horizontal, i.e., $\mathrm{v}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma\right) = 0_{TTM}$. This is the concept of *geodesic* or *autoparallel curve*.

Definition 4.5.5. Consider a connection on TM and the induced covariant derivative ∇. A **geodesic** of the connection is a curve $\gamma : [0,1] \to M$ such that

$$\nabla_{\frac{\partial}{\partial t}}(\hat{\gamma}) = 0_{TTM}, \tag{4.32}$$

where $\hat{\gamma} := (\gamma, \dot{\gamma}) : [0,1] \to TM$.

From the Example 4.5.1 we can see that, in local coordinates, the condition for a curve to be a geodesic is equivalent to satisfy the equation

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + N^i_j(x,y)\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0. \tag{4.33}$$

We can easily verify that the paths $\tau_M \circ \hat{\gamma} : [0,1] \to M$, where $\hat{\gamma} : [0,1] \to TM$ is integral curve of the semispray $\mathscr{X}_{\mathbb{H}} \in \mathfrak{X}_{\mathrm{ss}}(TM)$ associated with the connection, are geodesics of the connection.

Proposition 4.5.4. *Consider the connection $\mathbb{H}$, the semispray $\mathscr{X}_{\mathbb{H}} \in \mathfrak{X}_{ss}(TM)$ associated with the connection and an integral curve $\hat{\gamma} : [0,1] \to TM$ of $\mathscr{X}_H$. Then, $\gamma = \tau_M \circ \hat{\gamma}$ is geodesic of $\mathbb{H}$.*

Proof. Indeed, since $\mathscr{X}_{\mathbb{H}} = \mathrm{h}(\mathscr{X})$ for any semispray $\mathscr{X} \in \mathfrak{X}_{\mathrm{ss}}(TM)$. If $\hat{\gamma}$ is integral curve of $\mathscr{X}_{\mathbb{H}}$ then $\nabla_{\frac{\partial}{\partial t}}(\hat{\gamma}) = \mathrm{v} \circ \mathscr{X}_{\mathbb{H}} = \mathrm{v} \circ \mathrm{h} \circ \mathscr{X}$ and $\mathrm{v} \circ \mathrm{h} = 0$. □

The following proposition is consequence of the latter. It states the case for when the solutions of the Euler-Lagrange equations are geodesics of the connection determined by the canonical semispray. We need to recall from Proposition 3.1.4 that if the lagrangian is homogeneous of degree 2, then the canonical semispray is a spray.

Proposition 4.5.5. *Let $L : TM \to \mathbb{R}$ be a homogeneous lagrangian of degree 2 and $\mathscr{X}_o \in \mathfrak{X}_{ss}(TM)$ the canonical spray of (L, M). Then, the projections to M of the integral curves of $\mathscr{X}_o$ are geodesics of the connection $\mathbb{P} = [\mathbb{J}, \mathscr{X}_o]_{FN}$. In other words, the solutions of the EL equation are geodesics of $\mathbb{P}$.*

Proof. Given that the lagrangian function is homogeneous of degree 2, due to Proposition 3.1.4 we have that the canonical semispray $\mathscr{X}_o$ is a spray, this is, $\mathcal{L}_{\mathscr{V}}\mathscr{X}_o = [\mathscr{V}, \mathscr{X}_o] = \mathscr{X}_o$.

From Proposition 4.4.2 we know that the semispray $\mathscr{X}_{\mathbb{H}}$ associated with the connection $[\mathbb{J}, \mathscr{X}_o]$ is $\mathscr{X}_o$ if and only $\mathscr{X}_o$ is a spray, which is the case that concerns to us. Therefore, by using Proposition 4.5.4 we can assure that integral curves of the canonical spray $\mathscr{X}_o$ provide geodesics of $\mathbb{P} = [\mathbb{J}, \mathscr{X}_o]$ via their projection to M. □

Chapter 5

Constraints

In Chapter 3 we see that given a Lagrange space (M, L), any semispray can be constructed from the canonical semispray by adding vertical vector fields. In this chapter we take advantage of the Lagrange-d'Alembert principle to find semisprays such that their integral curves are resticted to a submanifold N of TM, i.e., we provide a mechanical system (M, L, ω) such that the semispray $\mathscr{X}$ satisfying $\Lambda(\mathscr{X}) = \omega$ is tangent to N.

The first section, *Nonholonomic constraints*, formulates the theory of constraints on the phase space TM in a way that we incorporate the constraint into a system as a set of forces, called *reaction forces*. In the section *Mechanical systems with constraints* we provide the method for treating constraints in mechanical systems. The last section, *Examples*, includes the *free particle* in $\mathbb{R}^3$ with a nonholonomic constraint and the *coin rolling without slipping* on $\mathbb{R}^2$.

5.1 Nonholonomic constraints

As is noted in the Introduction, nonholonomic constraints are *non-integrable* distributions on the configuration manifold M that are used as a restriction on the movement on the phase space TM of a given mechaical system. In this manner, the definition of nonholonomic constraint is independent of any other structure such as Lagrange spaces or symplectic forms.

Definition 5.1.1 ([Grifone and Mehdi, 1999])**.** Consider a manifold M. An **(admissible nonholonomic) constraint** is a codistribution $W \subset T^*TM$ such that

$$\dim W = \dim \mathbb{J}^* W. \tag{5.1}$$

Moreover,

a. The semibasic 1−forms $\omega \in \mathbb{J}^* W$ are called **reaction forces**.

b. The constraint $W \subset T^*TM$ is called **ideal constraint** if it annihilates the Liouville vector field $\mathscr{V} \in \mathfrak{X}(TM)$, i.e., $\alpha(\mathscr{V}) = 0$ for every $\alpha \in W$.

c. An **admissible semispray** for the constraint W is a semispray $\mathscr{X} \in \mathfrak{X}_{\text{ss}}(TM)$ such that $\alpha(\mathscr{X}) = 0$ for every $\alpha \in W$.

Note that condition (5.1) says that the number of independent reaction forces at every point is equal to the number of conditions defining the constraint, ([Vershik and Faddeev, 1995]). The following proposition characterizations of nonholonomic constraints in terms of a distribution $N \subset TM$. This allows to interpret constraints as a relation between coordinates of configurtation and of velocities. The $^\circ$ symbol represents the *annihilator*.

Proposition 5.1.1. *Let $N \subset TM$ be a subset of TM. The following statements are equivalent,*

a. $(TN)^\circ$ *is a nonholonomic constraint, i.e.,* $\dim(TN)^\circ = \dim \mathbb{J}^*(TN)^\circ$.

b. $(TN)^\circ$ *does not contains semibasic* $1-$*forms, i.e.,* $\mathbb{V}^\circ \cap (TN)^\circ = \{0\}$.

c. N is transversal to the vertical subbundle $\mathbb{V} \subset TTM$, i.e., $TN + \mathbb{V} = TTM$.

Proof. The equivalence between a. and b. is due to the fact that $\dim(TN)^\circ = \dim \mathbb{J}^*(TN)^\circ$. Therefore, the map $(TN)^\circ \to \mathbb{J}^*(TN)^\circ$ given by $\omega \mapsto \mathbb{J}^*\omega = \omega \circ \mathbb{J}$ is injective, the kernel of this map is $(TN)^\circ \cap \mathbb{V}^\circ$ and $\mathbb{V}^\circ = \Omega^1_{\text{sb}}(TM)$. The equivalence between c. and b. is seen by taking the annihilator on both sides of $TN + \mathbb{V} = TTM$, this is, $(TN)^\circ \cap \mathbb{V}^\circ = \{0\}$. □

Regarding $N \subset TM$ as a nonholonomic constraint we can rephrase definitions in 5.1.1 as follows,

a. The constraint $N \subset TM$ is *ideal* if the Liouville vector field $\mathscr{V} \in \mathfrak{X}(TM)$ is tangent to N, i.e., $\mathscr{V}|_N \in \mathfrak{X}(N)$.

b. A semispray $\mathscr{X} \in \mathfrak{X}_{\text{ss}}(TM)$ is *admissible* if $\mathscr{X}|_N \in \mathfrak{X}(N)$.

Remark 5.1.1. The condition for the constraint to be ideal expresses that the work of the reaction forces is zero (see Proposition 5.1.2 below).

Remark 5.1.2. The condition for a vector field to be an admissible semispray expresses that their integral curves project to curves on M such that their velocities are constrained to N.

The example given below provides an easy way to manage examples of mechanical systems with constraints. Here, the relations between configurations and velocities are given as functions $f_i : TM \to \mathbb{R}$ and we take as constraint the kernel of these functions.

Example 5.1.1. Consider a submersion $F = (f_1, \dots, f_{n-k}) : TM \to \mathbb{R}^{n-k}$, where $f_i \in C^\infty(TM)$. Then, consider the kernel of the submersion, $N := \ker F$. In this case, the annihilator of TN is the set of 1−forms

$$TN^\circ = \left\{\alpha \in \Omega^1(TM) \mid \alpha = \lambda_1 \mathrm{d}f_1 + \dots + \lambda_{n-k}\mathrm{d}f_{n-k},\ \lambda_i \in C^\infty(TM)\right\}.$$

Now, N (or the codistribution $(TN)^\circ$) is a nonholonomic constraint if $(TN)^\circ$ satisfies the condition (5.1), i.e., if the $n-k$ 1−forms

$$\mathrm{d}f_1 \circ \mathbb{J}\,, \mathrm{d}f_2 \circ \mathbb{J}\,, \dots,\ \mathrm{d}f_{n-k} \circ \mathbb{J}, \tag{5.2}$$

are linearly independent.

To finish this section we prove that reaction forces of ideal constraints do not work on curves subject to the constraint.

Proposition 5.1.2. *Let $N \in TM$ be an ideal nonholonomic constraint and $\omega \in \mathbb{J}^*(TN)^\circ$ a reaction force. Then, the work done by reaction forces trough curves in N is zero, i.e.,*

$$\int_{\hat{\gamma}} \omega = 0, \qquad \textit{for every} \quad \omega \in \mathbb{J}^*(TN)^\circ \quad \textit{and} \quad \hat{\gamma} : [0,1] \to N. \tag{5.3}$$

Proof. If the constraint $N \subset TM$ is ideal, then $\mathscr{V} \in TN$. Thus, for every $\alpha \in (TN)^\circ$ we have $\alpha(\mathscr{V}) = 0$. Reaction forces are of the form $\omega = \iota_{\mathbb{J}}\alpha$ for some $\alpha \in (TN)^\circ$. Then, for semisprays generating curves $\hat{\gamma} : [0,1] \to TN$, we have

$$\omega(\mathscr{X}) = \alpha \circ \mathbb{J}\mathscr{X} = \alpha(\mathscr{V}) = 0.$$

Therefore,

$$\int_{\hat{\gamma}} \omega = \int_0^1 \hat{\gamma}^*\omega = \int_0^1 \omega \circ \hat{\gamma}_* = 0.$$

□

5.2 Mechanical systems with constraints

We consider a *mechanical system with a nonholonomic constraint* as (M, L, ω, N), where (M, L) is a Lagrange space, $\omega \in \Omega^1_{\mathrm{sb}}(TM)$ a semibasic 1−form and $N := \ker F$ a nonholonomic constraint given by a function $F := (f_1, \dots, f_{n-k}) : TM \to \mathbb{R}^{n-k}$. In this case, there are another external forces in the system due to the constraint. We need to find a *physically acceptable* solution to the system, this is, we require an admissible semispray $\mathscr{X} \in \mathfrak{X}_{\mathrm{ss}}(TM)$ such that

$$\Lambda(\mathscr{X}) - \omega \in \mathbb{J}^*(TN)^\circ. \tag{5.4}$$

In fact, we can prove that there exists only one semispray satisfying this condition.

Proposition 5.2.1. *Let (M, L, ω, N) be a mechanical system with a nonholonomic constraint, where $N :=$ $\ker(F : TM \to \mathbb{R}^{n-k})$. Then, there exists a unique admissible semispray such that the force $\Lambda(\mathscr{X}) - \omega$ is a reaction force of the constraint, i.e., a semispray $\mathscr{X} \in \mathscr{X}_{ss}(TM)$ such that*

a. $\mathrm{d}F(\mathscr{X}) = 0$ *and* *(admissible semispray)*

b. $\Lambda(\mathscr{X}) - \omega \in \mathbb{J}^*(TN)^\circ$.

Proof. For a constraint given by a submersion $F := (f_1, \ldots, f_{n-k}) : TM \to \mathbb{R}^{n-k}$, reaction forces are of the form

$$\omega_\lambda = \lambda_1 \mathrm{d}f_1 \circ \mathbb{J} + \lambda_2 \mathrm{d}f_2 \circ \mathbb{J} + \cdots + \lambda_{n-k} \mathrm{d}f_{n-k} \circ \mathbb{J} \in \mathbb{J}^*(TN)^\circ,$$

where $\lambda_1, \ldots, \lambda_{n-k} \in C^\infty(TM)$. We know that for each $\omega_\lambda \in \mathbb{J}^*(TN)^\circ$ the vector field $\mathscr{X} \in \mathfrak{X}(TM)$ such that $\Lambda(\mathscr{X}) = \omega + \omega_\lambda$ exists and is a semispray (Proposition 3.2.2). Hence, we only have to prove there exists a unique reaction force $\omega_1 \in \mathbb{J}^*(TN)^\circ$ that makes the semispray admissible. Denote by $\mathscr{X}_i \in \mathfrak{X}(TM)$ each hamiltonian vector field

$$\imath_{\mathscr{X}_i} \Omega_L = -\mathrm{d}f_i.$$

Then, by formula A.9 we have that

$$\imath_{\mathbb{J}} \imath_{\mathscr{X}_i} \Omega_L = \imath_{\mathscr{X}_i} \imath_{\mathbb{J}} \Omega_L - \imath_{\mathbb{J}\mathscr{X}_i} \Omega_L = -\imath_{\mathbb{J}\mathscr{X}_i} \Omega_L = -\imath_{\mathbb{J}} \mathrm{d}f_i = -\mathrm{d}f_i \circ \mathbb{J}.$$

In this way, reaction forces are of the form

$$\omega_\lambda = \lambda_1 \Omega_L^\flat(\mathbb{J}\mathscr{X}_1) + \cdots + \lambda_{n-k} \Omega_L^\flat(\mathbb{J}\mathscr{X}_{n-k})$$

Let $\mathscr{X}_o \in \mathfrak{X}_{\rm ss}(TM)$ be the canonical semispray and $\mathscr{Y} \in \Gamma(\mathbb{V})$ the acceleration corresponding to the force ω. Thus, semisprays in the Lagrange-d'Alembert principle are

$$\mathscr{X}_\lambda = \mathscr{X}_o + \mathscr{Y} + \lambda_1 \mathbb{J}\mathscr{X}_1 + \cdots + \lambda_{n-k} \mathbb{J}\mathscr{X}_{n-k}.$$

The admissibility condition is $\mathrm{d}f_i(\mathscr{X}_\lambda) = 0$ for every $i =, \ldots, n-k$. Therefore, we have

$$\mathrm{d}f_i(\mathscr{X}_o + \mathscr{Y}) = -\lambda_j \mathrm{d}f_i(\mathbb{J}\mathscr{X}_j),$$

and replacing $\imath_{\mathscr{X}_i} \Omega_L = -\mathrm{d}f_i$ into this expression, we have

$$\Omega_L(\mathscr{X}_i, \mathbb{J}\mathscr{X}_j)\lambda_j = \mathrm{d}f_i(\mathscr{X}_o + \mathscr{Y}),$$

which is is a linear system for the λ's and has a unique solution due to the non-degeneracy of the 2−form Ω_L and the independence of the vector fields $\mathscr{X}_i$'s and $\mathscr{X}_j$'s. □

5.3 Examples

In this section we give examples of Lagrange spaces and nonholonomic constraints. First, we study the simplest case: *a free particle in the space with no constraints*. The configuration manifold in this case is $M := \mathbb{R}^3$ and the lagragian is just the kinetic energy. Next, following the same example, we include a nonholonomic and we find the admissible semispray in accordance with the Lagrange-d'Alembert principle.

The second example is *a coin on the table*, i.e., a vertical disk rolling without slipping over a plane. The configuration manifold is $\mathbb{R}^2 \times \mathbb{S} \times \mathbb{S}$. This example is taken from [Mladenova et al., 2014]. We study the Ehresmann connection determined by the solution of this example and their geodesics.

5.3.1 Free particle

The free particle is described just by the kinetic energy in the corresponding phase space. We work on the configuration manifold $M := \mathbb{R}^3$. The coordinates for the phase space TM are $(x_1, x_2, x_3, y_1, y_2, y_3) \in T\mathbb{R}^3$ (we lower the indices in order to not confuse with exponents). Therefore, the Lagrange space is (M, L), where

$$M = \mathbb{R}^3 \qquad \text{and} \qquad L = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right).$$

The lagrangian 2−form To calculate the lagrangian 2−form, first compute the lagrangian 1− form

$$\Theta_L = \mathcal{L}_{\mathbb{J}} L = \mathrm{d}L \circ \mathbb{J} = \left(y_1 \mathrm{d}y_1 + y_2 \mathrm{d}y_2 + y_3 \mathrm{d}y_3\right) \circ \mathbb{J}. \tag{5.5}$$

The lagrangian 2−form is the differential of Θ_L, so we have

$$\Omega_L = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2 + \mathrm{d}y_3 \wedge \mathrm{d}x_3. \tag{5.6}$$

The canonical semispray Observe that the lagrangian function is homogeneous of degree 2, therefore, by Proposition 3.1.4 we know that the canonical semispray is a spray. Moreover, also by the homogeneity of the lagrangian, the energy function E_L equals the lagrangian. Hence, the canonical spray of (M, L) is the hamiltonian vector field $\mathscr{X}_o \in \mathfrak{X}(T\mathbb{R}^3)$ associated with L, i.e., $\mathscr{X}_o$ satisfies

$$\imath_{\mathscr{X}_o}\Omega_L = -\mathrm{d}L. \tag{5.7}$$

In coordinates, writing the spray as $\mathscr{X}_o = (x, y; y, G(x, y))$, the this expression is

$$-y_1\mathrm{d}y_1 - y_2\mathrm{d}y_2 - y_3\mathrm{d}y_3 - 2G^1\mathrm{d}x_1 - 2G^2\mathrm{d}x_2 - 2G^3\mathrm{d}x_3 = -(y_1\mathrm{d}y_1 + y_2\mathrm{d}y_2 + y_3\mathrm{d}y_3). \tag{5.8}$$

Therefore, all the coefficients of the spray are zero and the spray is simply

$$\mathscr{X}_o = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3}. \tag{5.9}$$

The Lagrangian connection Since all the coefficients of the lagrangian are zero, the coefficients of the connection determined by the spray are zero. Therefore, the horizontal subbundle is $\mathbb{H} := \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ and it is lagrangian with respecto to (5.6).

Nonholonomic constraint We introduce a nonholonomic constraint to the dynamics of the free particle. This is done with a function $f : TM \to \mathbb{R}$ given by

$$f(x_1, x_2, x_3, y_1, y_2, y_3) := y_3 - x_2 y_1. \tag{5.10}$$

The particle is restricted to move on $\ker f$. The constraint codistribution is $W := \left\{\alpha \in \Omega^1 \mid \alpha = \lambda \mathrm{d}f\right\}$ and since $\mathrm{d}f \circ \mathbb{J} = \mathrm{d}x_3 - x_2 \mathrm{d}x_1$ we verify that $\dim W = \dim \mathbb{J}^* W$.

Lagrange-d'Alembert Principle We look for an admissible semispray to the constraint satisfying the Lagrange-d'Alembert principle for a reaction force. Reaction forces are of the form $\omega_\lambda = \lambda(\mathrm{d}x_3 - x_2 \mathrm{d}x_1)$. Then, from the Lagrange-d'Alembert principle, the trajectories of a mechanical system (M, L, ω_λ) are integral curves of the semispray $\mathscr{X}_\lambda = \mathscr{X}_o + \mathscr{Y}_\lambda \in \mathfrak{X}_{\mathrm{ss}}(TM)$ (where $\mathscr{Y}_\lambda \in \Gamma(\mathbb{V})$) satisfying

$$\Lambda(\mathscr{X}) = \Lambda(\mathscr{X}_o) + \Omega_L^\flat(\mathscr{Y}_\lambda) = \omega_\lambda. \tag{5.11}$$

Write the vertical vector field as $\mathscr{Y}_\lambda = (x_i, y_i, 0_{TM}, -2G_\lambda^i(x, y))$, then the Lagrange-d'Alembert principle in coordinates is

$$-2G_\lambda^1 \mathrm{d}x_1 - 2G_\lambda^2 \mathrm{d}x_2 - 2G_\lambda^3 \mathrm{d}x_3 = \lambda \mathrm{d}x_3 - \lambda x_2 \mathrm{d}x_1. \tag{5.12}$$

The admissible semispray The coefficients of the semisprays satisfying the Lagrange-d'Alembert principle are of the form

$$2G_\lambda^1(x, y) = \lambda x_2, \qquad 2G_\lambda^2(x, y) = 0 \qquad \text{and} \qquad 2G_\lambda^3(x, y) = \lambda. \tag{5.13}$$

Moreover, an admissible semispray has to saistify $\mathrm{d}f(\mathscr{X}_\lambda) = 0$, i.e,

$$\left(\mathrm{d}y_3 - x_2 \mathrm{d}y_1 - y_1 \mathrm{d}x_2\right)\left(x_i \frac{\partial}{\partial x_i} - 2G^i(x, y)\frac{\partial}{\partial y_i}\right) = -y_1 x_2 + 2\lambda(x_2)^2 - 2\lambda = 0. \tag{5.14}$$

Then, the value of λ for the admissible semispray is $\lambda = \frac{1}{2}\frac{y_1 x_2}{((x_2)^2 - 1)}$, with $(x_2)^2 \neq 1$, and the admissible semispray is

$$\mathscr{X} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3} - \frac{y_1 (x_2)^2}{((x_2)^2 - 1)}\frac{\partial}{\partial y_1} + \frac{y_1 x_2}{((x_2)^2 - 1)}\frac{\partial}{\partial y_3} \tag{5.15}$$

5.3.2 A coin on the table

Consider a circular thin disk (imagine a coin of negligible thickness) rolling over a plane *without slipping* (the velocity at the contact point of the coin with the plain is zero). In this example the configuration manifold is $M := \mathbb{R}^2 \times \mathbb{S} \times \mathbb{S}$. The $\mathbb{R}^2$ factor represents the plane; the first $\mathbb{S}$, the *precession* of the disk and the second $\mathbb{S}$, the *spin*. Then, we denote the coordinates of TM as:

$$TM = (\mathbb{R}^2 \times \mathbb{S} \times \mathbb{S}) \times (\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}) \ni (x_1, x_2, \phi, \psi; y_1, y_2, y_\phi, y_\psi). \tag{5.16}$$

Note that the velocity of the contact point is the sum of the velocity of the center of the disk (y_1, y_2) and the velocity at the base point of the disk $(-y_\psi \cos\phi, -y_\psi \sin\phi)$. Thus, the condition of rolling without slipping provides the following conditions

$$y_1 - y_\psi \cos\phi = 0 \qquad \text{and} \qquad y_2 - y_\psi \sin\phi = 0. \tag{5.17}$$

The Lagrange space To construct our Lagrange space, we need a lagrangian function, this is the kinetic energy $L : TM \to \mathbb{R}$ of the system,

$$L := \frac{1}{2}(y_1^2 + y_2^2 + I y_\phi^2 + J y_\psi^2). \tag{5.18}$$

where I and J are the moments of inertia.

The lagrangian $2-$**form** To define the symplectic manifold (TM, Ω_L) with Ω_L the lagrangian $2-$form corresponding to (M, L), calculate the lagrangian $1-$form,

$$\Theta_L = \mathcal{L}_{\mathbb{J}} L = \mathrm{d}L \circ \mathbb{J} = \left(y_1 \mathrm{d}y_1 + y_2 \mathrm{d}y_2 + I y_\phi \mathrm{d}y_\phi + J y_\psi \mathrm{d}y_\psi\right) \circ \mathbb{J} = y_1 \mathrm{d}x_1 + y_2 \mathrm{d}x_2 + I y_\phi \mathrm{d}\phi + J y_\psi \mathrm{d}\psi. \tag{5.19}$$

Then, the lagrangian $2-$form is

$$\Omega_L = \mathrm{d}\Theta_L = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2 + I \mathrm{d}y_\phi \wedge \mathrm{d}\phi + J \mathrm{d}y_\psi \wedge \mathrm{d}\psi. \tag{5.20}$$

The canonical semispray As in the example of the free particle, we have that the canonical semispray is a spray since the lagrangian function is homogeneous of degree 2.The canonical spray is

$$\mathscr{X}_0 = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_\phi \frac{\partial}{\partial \phi} + y_\psi \frac{\partial}{\partial \psi}.$$

The nonholonomic constraint We see that the conditions 5.17 define a nonholonomic constraint. Consider the function $F := (f_1, f_2) : TM \to \mathbb{R}$ given by

$$f_1 = y_1 - y_\psi \cos\phi \quad \text{and} \quad f_2 = y_2 - y_\psi \sin\phi. \tag{5.21}$$

Then, $N := \ker F \subset TM$ is a nonholonomic constraint. Indeed, we see that

$$(TN)^\circ = \left\{\alpha \in \Omega^1(TM) \mid \alpha = \lambda_1 \mathrm{d}f_1 + \lambda_2 \mathrm{d}f_2\right\},$$

where $\mathrm{d}f_1 = \mathrm{d}y_1 - \cos\phi \mathrm{d}y_\psi + y_\psi \sin\phi \mathrm{d}\phi$ and $\mathrm{d}f_2 = \mathrm{d}y_2 - \sin\phi \mathrm{d}y_\psi - y_\psi \cos\phi \mathrm{d}\phi$. Hence, the reaction forces are

$$\mathbb{J}^*(TN)^\circ = \left\{\omega_\lambda \in \Omega^1(TM) \mid \omega_\lambda = \lambda_1(\mathrm{d}x_1 - \cos\phi \mathrm{d}\psi) + \lambda_2(\mathrm{d}x_2 - \sin\phi \mathrm{d}\psi)\right\},$$

and $\dim(TN)^\circ = \dim \mathbb{J}^*(TN)^\circ$. Therefore, by Proposition 5.1.1, $N \subset TM$ is a nonholonomic constraint.

The Lagrange-d'Alembert principle If we regard each reaction force $\omega_\lambda \in \mathbb{J}^*(TN)^\circ$ as an external force, then the Lagrange-d'Alembert principle asserts that trajectories of the mechanical system are solutions of the semispray $\mathscr{X}_\lambda \in \mathfrak{X}(TM)$ such that

$$\iota_{\mathscr{X}_\lambda}\Omega + \mathrm{d}E = \omega_\lambda,$$

where $\omega = \lambda_1(\mathrm{d}x_1 - \cos\phi \mathrm{d}\psi) + \lambda_2(\mathrm{d}y_2 - \sin\phi \mathrm{d}\psi)$. Thus, for each $\omega_\lambda \in \mathbb{J}^*(TN)^\circ$, the coefficients of the semispray $\mathscr{X}_\lambda \in \mathfrak{X}_{\mathrm{ss}}(TM)$ are

$$-2G^1_\lambda(x,y) = \lambda_1, \quad -2G^2_\lambda(x,y) = \lambda_2, \quad -2G^\phi_\lambda(x,y) = 0 \quad \text{and} \quad -2G^\psi_\lambda(x,y) = -\frac{1}{J}(\lambda_1\cos\phi + \lambda_2 \sin\phi). \tag{5.22}$$

The semispray is

$$\mathscr{X}_\lambda = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_\phi\frac{\partial}{\partial \phi} + y_\psi\frac{\partial}{\partial \psi} + \lambda_1\frac{\partial}{\partial y_1} + \lambda_2\frac{\partial}{\partial y_2} - \frac{1}{J}(\lambda_1\cos\phi + \lambda_2\sin\phi)\frac{\partial}{\partial y_\psi}.$$

Admissible semispray We look for an admissible semispray due to its physical significance: their integral curves are constrained to N (Remark 5.1.2). If $\mathscr{X}_\lambda$ is an admissible semispray then

$$\mathrm{d}f_1(\mathscr{X}_\lambda) = 0 \quad \text{and} \quad \mathrm{d}f_2(\mathscr{X}_\lambda) = 0,$$

which implies

$$\lambda_1 + \frac{1}{J}(\lambda_1\cos\phi + \lambda_2\sin\phi)\cos\phi + y_\phi y_\psi \sin\phi = 0$$
$$\text{and} \quad \lambda_2 + \frac{1}{J}(\lambda_1\cos\phi + \lambda_2\sin\phi)\sin\phi - y_\phi y_\psi\cos\phi = 0.$$

Then, the solution for the functions $\lambda_1, \lambda_2 \in C^\infty(TM)$ is

$$\lambda_1 = -y_\phi y_\psi \sin\phi \qquad \text{and} \qquad \lambda_2 = y_\phi y_\psi \cos\phi. \tag{5.23}$$

Therefore, the admissible semispray is

$$\mathscr{X} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_\phi \frac{\partial}{\partial \phi} + y_\psi \frac{\partial}{\partial \psi} - y_\phi y_\psi \sin\phi \frac{\partial}{\partial y_1} + y_\phi y_\psi \cos\phi \frac{\partial}{\partial y_2}.$$

We can write the admissible semispray in the form of equations of motion,

$$\dot{x}_1 = y_1, \qquad \dot{y}_1 = -y_\phi y_\psi \sin\phi, \tag{5.24}$$

$$\dot{x}_2 = y_2, \qquad \dot{y}_2 = y_\phi y_\psi \cos\phi \tag{5.25}$$

$$\dot{\psi} = y_\psi, \qquad \dot{y}_\psi = 0, \tag{5.26}$$

$$\dot{\phi} = y_\phi, \qquad \dot{y}_\phi = 0 \tag{5.27}$$

which are the equations found in [Bloch et al., 1996] for this example. The solutions to this system of equations are the trajectories of the mechanical system.

5.3.3 Connections associated to the semisprays

Now we study the connections related with the example of the coin on the table. For example, the canonical semispray of the Lagrange space of the coin is

$$\mathscr{X}_0 = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_\phi \frac{\partial}{\partial \phi} + y_\psi \frac{\partial}{\partial \psi}. \tag{5.28}$$

Evidently, the nonlinear connection $\mathbb{P} = [\mathbb{J}, \mathscr{X}_0]_{FN}$ determined by $\mathscr{X}_o$ has as semispray associated the same $\mathscr{X}_o$ since X_o is homogeneous of degree 2, i.e, $\mathrm{h}(\mathscr{X}_o) = \mathscr{X}_o$. The coefficients of this nonlinear connection are just $N^i_j = 0$, hence, the horizontal projector is just $\mathrm{h} = \mathrm{d}x_i \oplus \frac{\partial}{\partial x_i}$ (take $x_3 = \phi$ and $x_4 = \psi$). Since the lagrange 2−form is $\Omega_L = \mathrm{d}y_i \wedge \mathrm{d}x_i$ it follows that this connection is a lagrangian connection, i.e., $\Omega_L(\mathrm{h}\mathscr{X}, \mathrm{h}\mathscr{Y}) = 0$ for every $\mathscr{X}, \mathscr{Y} \in \mathfrak{X}(TM)$.

The admissible semispray resulting from the Lagrange-d'Alembert principle is more interesting since in includes the acceleration due to the reaction force of the constraint, we found it to be

$$\mathscr{X} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_\psi \frac{\partial}{\partial \psi} + y_\phi \frac{\partial}{\partial \phi} - y_\phi y_\psi \sin\phi \frac{\partial}{\partial y_1} + y_\phi y_\psi \cos\psi \frac{\partial}{\partial y_2}. \tag{5.29}$$

To find the coefficients of the connection determined by this semispray we have to use the coefficients of the semispray $G^i \in C^\infty(TM)$, which are

$$G^1(x,y) = \frac{1}{2} y_\psi y_\phi \sin\psi, \quad G^2(x,y) = -\frac{1}{2} y_\phi y_\psi \cos\phi, \quad G^\phi(x,y) = 0 \quad \text{and} \quad G^\psi(x,y) = 0.$$

Now, using Equation (4.16):

$$N^i_j(x,y) = \frac{\partial G^i(x,y)}{\partial y^j},$$

we calculate the coefficients of the connection.

$$\begin{aligned}
&N^1_1 = 0, && N^1_2 = 0, && N^1_\phi = \frac{1}{2} y_\psi \sin\phi, && N^1_\psi = \frac{1}{2} y_\phi \sin\phi,\\
&N^2_1 = 0, && N^2_2 = 0, && N^2_\phi = -\frac{1}{2} y_\psi \cos\phi, && N^2_\psi = -\frac{1}{2} y_\phi \cos\phi,\\
&N^\phi_1 = 0, && N^\phi_2 = 0, && N^\phi_\phi = 0, && N^\phi_\psi = 0,\\
&N^\psi_1 = 0, && N^\psi_2 = 0, && N^\psi_\phi = 0, && N^4_\psi = 0.
\end{aligned}$$

From Equation (4.33), geodesics satisfy

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + N^i_j(x,y)\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0. \tag{5.30}$$

With the notation $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \dot{y}^i$ and $\frac{\mathrm{d}x^i}{\mathrm{d}t} = \dot{x}^i$ we get the set of equations:

$$\begin{aligned}
\dot{y}_1 + N^1_j(x,y)\dot{x}^j &= \dot{y}_1 + \frac{1}{2} y_\psi \sin\phi\dot{\phi} + \frac{1}{2} y_\phi \sin\phi\dot{\psi},\\
\dot{y}_2 + N^2_j(x,y)\dot{x}^j &= \dot{y}_2 - \frac{1}{2} y_\psi \cos\phi\dot{\phi} - \frac{1}{2} y_\phi \cos\phi\dot{\psi},\\
\dot{y}_\phi + N^\phi_j(x,y)\dot{x}^j &= \dot{y}_\phi,\\
\dot{y}_\psi + N^\psi_j(x,y)\dot{x}^j &= \dot{y}_\psi.
\end{aligned}$$

Then, replacing the equations of motion (5.24), (5.25), (5.26) and (5.27) we get the desired result:

$$\begin{aligned}
\dot{y}_1 + N^1_j(x,y)\dot{x}^j &= 0,\\
\dot{y}_2 + N^2_j(x,y)\dot{x}^j &= 0,\\
\dot{y}_\psi + N^\phi_j(x,y)\dot{x}^j &= 0,\\
\dot{y}_\phi + N^\psi_j(x,y)\dot{x}^j &= 0.
\end{aligned}$$

Conclusion of the example We have obtained the motion of a vertical disk rolling on a plane. This is, for the configuration manifold $M = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ and lagrangian function $L = \frac{1}{2}(y_1^2 + y_2^2 + I y_\phi^2 + J y_\psi^2)$ we have constructed a mechanical system (M, L, ω) such that its trajectories on TM are restricted to $N \subset TM$, where N is determined by the constraints:

$$y_1 - y_\psi \cos\phi = 0, \quad \text{and} \quad y_2 - y_\psi \sin\phi = 0.$$

Moreover, we found that these trajectories are the velocities at each point of the geodesics of the connection determined by the admissible semispray semispray satisfying the Lagrange-d'Alembert principle.

Appendices

Appendix A

Frölicher-Nijenhuis calculus

In this appendix we present the required calculus of differential forms used in the thesis. The reference where proofs of the unproved facts we enunciate here are found is [Michor, 2008]. Besides, we mention particular cases of some formulas that are recurrent along the thesis.

A.1 Derivation and the algebra of differential forms

Consider in a manifold M, the graded algebra of differential forms:

$$\Omega^\bullet(M) = \bigoplus_{l=-\infty}^{\infty} \Omega^l(M).$$

Definition A.1.1. A **graded derivation** of degree k is a linear map:

$$D : \Omega^\bullet(M) \to \Omega^{\bullet+k}(M),$$

such that:

$$D(\phi \wedge \psi) = D(\phi) \wedge \psi + (-1)^{kl} \phi \wedge D(\psi),$$

where $\phi \in \Omega^l(M)$. We denote by $\mathrm{Der}_k(M)$ the space of all graded derivations of degree k.

Example A.1.1. The exterior derivative d is a graded derivation of degree 1. Let $\mathscr{X} \in \mathfrak{X}(M)$ be a vector field. Then, the insertion $\iota_{\mathscr{X}}$ of $\mathscr{X}$ into a differential form is a graded derivation of degree -1. The Lie derivative $\mathcal{L}_{\mathscr{X}}$ along $\mathscr{X}$ is a graded derivation of degree 0.

A.1.1 Algebraic derivation

Algebraic derivations generalize the notion of the insertion $\iota_{\mathscr{X}} \in \mathrm{Der}_{-1}(M)$ of vector fields $\mathscr{X} \in \mathfrak{X}(M)$.

Definition A.1.2. A derivation $D \in \mathrm{Der}_K(M)$ is called **algebraic** if

$$D|_{\Omega^0(M)} = 0,$$

i.e., $D(f) = 0$ for every $f \in C^\infty(M)$.

Since derivations are determined by their action on 0−forms and 1−forms, it follows that an algebraic derivation are determined by its restriction to 1−forms. If $D_x \in \mathrm{Der}_k(\bigwedge T_x^* M)$ is algebraic, then $D_x|_{T_x^*M} : T_x^* M \to \bigwedge^{k+1} T^* M$ may be viewed as an element $K_x \in \bigwedge^{k+1} T_x^* M \oplus T_x M$. To express this dependence we write:

$$D = \iota_K, \quad \text{where} \quad K \in \Gamma(\bigwedge^{k+1} T^* M \oplus TM).$$

Denote the space of **vector valued k−forms** as

$$\Omega^k(M, TM) := \Gamma(\bigwedge^{k+1} T^* M \oplus TM),$$

and the **space of vector valued differential forms**

$$\Omega^\bullet(M, TM) := \bigoplus_{l=0}^{\dim M} \Omega^l(M, TM).$$

Summarizing this discussion we have:

Theorem A.1.1. *Let $K \in \Omega^{k+1}(M, TM)$ and $\omega \in \Omega^l(M)$. Then, for any $\mathscr{X}_1, \dots, \mathscr{X}_{k+l} \in \mathfrak{X}(M)$, we have that*

$$\iota_K \omega(\mathscr{X}_1, \dots, \mathscr{X}_{k+l}) = \frac{1}{(k+1)!(l-1)!} \sum_{\sigma \in S_{k+l}} \mathrm{sign}\sigma \; \omega\big(K(\mathscr{X}_{\sigma(1)}, \dots, \mathscr{X}_{\sigma(k+1)}), \mathscr{X}_{\sigma(k+2)}, \dots, \mathscr{X}_{\sigma(k+l)}\big) \tag{A.1}$$

is and algebraic derivation of degree k and any algebraic derivation is of this form.

A.1.2 Lie derivation

Lie derivations generalize the usual definition of Lie derivative $\mathcal{L}_{\mathscr{X}} \in \mathrm{Der}_0(M)$ along a vector field $\mathscr{X} \in \mathfrak{X}(M)$.

Lemma A.1.1. *Let $D_k \in Der_k(M)$ and $D_l \in Der_l(M)$. Then, the graded commutator*

$$[D_l, D_k] := D_k \circ D_l - (-1)^{kl} D_l \circ D_k$$

is a graded derivation of degree $k + l$.

Definition A.1.3. Let $K \in \Omega^k(M, TM)$, the **Lie derivative** $\mathcal{L}_K \in \mathrm{Der}_k(M)$ is defined by

$$\mathcal{L}_K := [\imath_K, \mathrm{d}] = \imath_K \mathrm{d} - (-1)^{k-1} \mathrm{d} \imath_K, \tag{A.2}$$

where $\mathrm{d} \in \mathrm{Der}_1(M)$ is the exterior derivative.

Theorem A.1.2 (Frölicher-Nijenhuis decomposition)**.** *Let $D \in \mathrm{Der}_k(M)$. Then, there are unique $K \in \Omega^k(M, TM)$ and $L \in \Omega^{k+1}(M, TM)$ such that*

$$D = \mathcal{L}_K + \imath_L. \tag{A.3}$$

Corollary A.1.1. *A derivation $D = \mathcal{L}_K + \imath_L$ is algebraic if and only if $K = 0$. Moreover, a derivation $D = \mathcal{L}_K + \imath_L$ satisfies $[D, \mathrm{d}] = 0$ if and only if $L = 0$.*

Corollary A.1.2. *If $\mathscr{X} \in \mathfrak{X}_{ss}(TM)$ is a semispray and $\mathbb{J} \in \Omega^1(TM, TTM)$ the vertical endomorphism, then:*

$$[\imath_{\mathscr{X}}, \imath_{\mathbb{J}}] = \imath_{\mathscr{X}} \circ \imath_{\mathbb{J}} - \imath_{\mathbb{J}} \circ \imath_{\mathscr{X}} = \imath_{\mathscr{V}} \tag{A.4}$$

$$[\imath_{\mathscr{V}}, \mathcal{L}_{\mathbb{J}}] = \imath_{\mathscr{V}} \mathcal{L}_{\mathbb{J}} + \mathcal{L}_{\mathbb{J}} \imath_{\mathscr{V}} = \imath_{\mathbb{J}} \tag{A.5}$$

A.2 Frölicher-Nijenhuis bracket

Note that if $\mathcal{L}_K \in \mathrm{Der}_k(M)$ and $\mathcal{L}_L \in \mathrm{Der}_l(M)$ are derivation commuting with the exterior derivative d, then we have that

$$[[\mathcal{L}_K, \mathcal{L}_L], \mathrm{d}] = 0.$$

Therefore, by the Frölicher-Nijenhuis decomposition theorem there exists a vector-valued form $[K, L]_{FN} \in \Omega^{k+l}(M)$ such that we can write the commutator $[\mathcal{L}_K, \mathcal{L}_L]$ as

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]_{\mathrm{FN}}}.$$

This vector-valued 1−form is called the *Frölicher-Nijenhuis bracket* of K and L.

Definition A.2.1. The **Frölicher-Nijenhuis bracket** is the unique vector valued differential form

$$[\cdot,\cdot]_{\mathrm{FN}} : \Omega^k(M, TM) \times \Omega^l(M, TM) \to \Omega^{k+l} \ : \ (K, L) \mapsto [K, L]_{\mathrm{FN}}, \tag{A.6}$$

such that

$$\mathcal{L}_{[K,L]_{\mathrm{FN}}} = [\mathcal{L}_K, \mathcal{L}_L]. \tag{A.7}$$

Proposition A.2.1. *If $K \in \Omega^0(M, TM) = \mathfrak{X}(M)$ and Ł $\in \Omega^1(M, TM)$, then for any vector field $\mathscr{X} \in \mathfrak{X}(M)$ we have*

$$[K, L]_{FN}(\mathscr{X}) = [K, L\mathscr{X}] - L[K, \mathscr{X}]. \tag{A.8}$$

Proposition A.2.2. *If $K, L \in \Omega^1(M, TM)$, then for any vector fields $\mathscr{X}, \mathscr{Y} \in \mathfrak{X}(M)$, we have*

$$\begin{aligned}[K, L]_{FN}(\mathscr{X}, \mathscr{Y}) = &[K\mathscr{X}, L\mathscr{Y}] + [L\mathscr{X}, K\mathscr{Y}] + (K \circ L + L \circ K)[\mathscr{X}, \mathscr{Y}] \\ &- K([L\mathscr{X}, \mathscr{Y}] + [\mathscr{X}, L\mathscr{Y}]) - L([K\mathscr{X}, \mathscr{Y}] + [\mathscr{X}, K\mathscr{Y}]).\end{aligned}$$

Proposition A.2.3. *Let $K \in \Omega^1(M, TM)$ and $\mathscr{X} \in \mathfrak{X}(M)$, then:*

$$[\imath_{\mathscr{X}}, \imath_K] = \imath_{\mathscr{X}} \circ \imath_K - \imath_K \circ \imath_{\mathscr{X}} = \imath_{K\mathscr{X}}, \tag{A.9}$$

$$[\imath_{\mathscr{X}}, \mathcal{L}_K] = \imath_{\mathscr{X}} \circ \mathcal{L}_K + \mathcal{L}_K \circ \imath_{\mathscr{X}} = \mathcal{L}_{K\mathscr{X}} + \imath_{[K, \mathscr{X}]_{FN}}. \tag{A.10}$$

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