The concept of monodromy for linear problems and its application

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Introduction

The notion of a monodromy matrix (operator) naturally appears under the study of linear systems with periodic coefficients. This notion gives rise to the well known result [3, 12] on the reducibility at the linear periodic systems (Floquet's Theorem) which says that the monodromy matrix contains the complete information about a given system.

The goal of the present work is to develop a unified concept of monodromy for linear systems on Lie algebras in the quasiperiodic and decreasing cases. The quasiperiodic case is a natural generalization of the periodic one. The decreasing case can be interpreted as the "limiting" periodic case (the period tends to infinity). Such class of linear problems arise in the integrability theory of nonlinear partial differential equations in the frame work of the so-called inverse scattering method [7, 8].

The main idea of this method is to represent a nonlinear partial differential equation as the consistence condition for two linear problems which leads to the study of the zero curvature equation [7], or Lax equation [8]. The important feature of the inverse scattering method is that, the linear problem involves a (spectral) parameter λ. The main idea is to study the analytic properties of the fundamental solution and the monodromy in λ. This gives us a spectral (scattering) data determining the problem. The time evolution of the spectral data gives solutions to the original nonlinear evolutory equation.

In Chapter 1-3, we study the linear problem and the zero curvature equation in the quasiperiodic and decreasing cases (independtly of the integrability theory for nonlinear equations) and then, in Chapter 4 we illustrate our result for the case of nonlinear Schrödinger equation [7].

In chapter 1, the goal is to give a definition of the monodromy for linear system on general matrix Lie algebras in the quasiperiodic and rapidly decreasing cases. We describe general properties of the fundamental solution and the monodromy matrix and study the dependence of the "spectral" parameter.

In the next chapter we apply the general results obtained in Chapter 1 to the study of linear problems on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ possessing the involution property. Such a class of linear problems plays an important role in the integrability theory for nonlinear evolution equations.

In Chapter 3, we formulate results on the inverse problem for linear systems on $\mathfrak{sl}(2, \mathbb{C})$ in the rapidly decreasing case and zero curvature equation. We mean that we shall show that it is possible reconstruct the linear system in $\mathfrak{sl}(2, \mathbb{C})$ (Chapter 2) from its spectral data (definition (2.2.2)) and assuming the linear time dynamics for the spectral data (2.2.91),(2.2.92) we shall get solution for the zero curvature equation.

Finally, as an application of the inverse problem, we construct some solutions of the Nonlinear Schrödinger Equation (NLS equation). This equation arises in various physical contexts, for example, it describes the effects of self-focusing of the envelope of a monochromatic plane wave propagating in nonlinear media [2]. The NLS equation appears also in the theory of surfaces waves on shallow water [4]. Equation (3.3.1) may be also considered as the Hatree-Fock equation for one dimensional quantum Boson gas equation with point intersection . Physically, the constant κ in (3.3.1) plays the role of acoupling constant: the case $\kappa > 0$ corresponds to attractive interaction and $\kappa < 0$ is the repulsive case. The two cases are essentially different in optical applications, describing self-focusing or defocusing of the light rays [2]. Mathematically, these two cases are also very different because the first one correspond to a selfadjoint linear problem while the second one is related a non-selfadjoint linear problem. The nonlinear Schrödinger equation was first solved by the inverse scattering method by Zakharov and Shabat [13]. In our treatment we shall follow an approach [7], using the result of Chapter 3. In the context of the integrability of NLS equation, the key observation is that, the NLS equation admits a zero curvature representation or Lax's pair.

Chapter 1

Monodromy for linear system with boundary conditions.

The goal of this chapter is to give a definition of the monodromy for linear system on general matrix Lie algebras in the quasiperiodic and rapidly decreasing cases. We describe general properties of the fundamental solution and the monodromy matrix and study the dependence of the "spectral" parameter.

1.1 Fundamental solutions

Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathfrak{gl}(\mathcal{V})$ the Lie algebra of all the linear transformation of $\mathcal{V}$ and by $\mathbf{GL}(\mathcal{V})$ the **general linear group** consisting of all invertible linear transformation.

Given a C^∞ linear function $\mathbb{R} \ni x \mapsto \mathbf{U}(x) \in \mathfrak{gl}(\mathcal{V})$, consider the follows linear system

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f}, \quad (\mathbf{f} = \mathbf{f}(x) \in \mathcal{V}) \tag{1.1.1}$$

We shall assume that $\mathbf{U}$ is bounded in $\mathbb{R}$ with respect to some norm on $\mathfrak{gl}(\mathcal{V})$

$$\|\mathbf{U}(x)\| < \infty, \quad \text{on} \quad \mathbb{R}. \tag{1.1.2}$$

Then, as is well known, [10, 11], there exists the **fundamental solution** of (1.1.1), that is, a function $\mathbb{R}^2 \ni (x, y) \mapsto \mathbf{F}(x, y)$ satisfying the Cauchy Problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x, y) = \mathbf{U}(x)\mathbf{F}(x, y), \tag{1.1.3}$$

$$\mathbf{F}(x, y)|_{x=y} = \mathbf{I}, \tag{1.1.4}$$

for every $y \in \mathbb{R}$. The solution of (1.1.1) with initial data $\mathbf{f}|_{x=y} = f^0$ is given by

$$\mathbf{f}(x) = \mathbf{F}(x, y)f^0. \tag{1.1.5}$$

For a fixed $y \in \mathbb{R}$.

Proposition 1.1.1. *The fundamental solution $\mathbf{F}(x, y)$ is differentiable in x, y and has the properties:*

(i) Non degeneracy,

$$\det \mathbf{F}(x, y) \neq 0., \tag{1.1.6}$$

and hence $\mathbf{F}(x, y) \in \mathbf{GL}(\mathcal{V})$.

(ii) The transition property

$$\mathbf{F}(x,z)\mathbf{F}(z,y) = \mathbf{F}(x,y), \tag{1.1.7}$$

for all x, y z.

(iii) The inverse of the monodromy matrix is given by

$$\mathbf{F}^{-1}(x,y) = \mathbf{F}(y,x),$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbf{F}(x,y) = -\mathbf{F}(x,y)\mathbf{U}(y).$$

Proof. (i) Let, $s \in \mathbb{R}$ be a fixed point. The fundamental solution $\mathbf{F}(x,y)$ can be seen as an integrable curve on the differentiable manifold $\mathfrak{gl}(V)$, so the fundamental solution $\mathbf{F}(x,y)$ is a continuous map that joints the identity element $\mathbf{I}$, with the point $\mathbf{F}(s,y)$. Therefore, for each x the fundamental solution $\mathbf{F}(x,y)$ is in the connected component of the identity element $\mathfrak{gl}^0(V)$. Since the determinant of a matrix is a continuous function

$$\mathbf{GL}^0(V) = \{X \in \mathbf{GL}(V) |\, \det X > 0\},$$

and therefore, $\mathbf{F}(x,y)$ is in $\mathbf{GL}(V)$.

ii) Consider that the points z and y are fixed, recall that $\mathbf{F}(x,y)$ is the fundamental solution of the linear problem (1.1.3),(1.1.4); if we change the boundary condition at $x = z$, then $\mathbf{F}|_{x=z} = \mathbf{F}(z,y)$. On the other hand, the matrix function $\mathbf{G}(x,y) = \mathbf{F}(x,z)\mathbf{F}(z,y)$ also satisfy the linear problem and the same boundary condition, therefore by uniqueness $\mathbf{F}(x,z)\mathbf{F}(z,y) = \mathbf{F}(x,y)$, as we desired.

iii) By *ii*) we have

$$\mathbf{F}(x,y)\mathbf{F}(y,x) = \mathbf{F}(x,x) = \mathbf{I}$$

then $\mathbf{F}^{-1}(x,y) = \mathbf{F}(y,x)$.

iv) By (ii), we know that $\mathbf{F}(x,y)\mathbf{F}(y,x) = \mathbf{I}$. From this, we can derive

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{F}(x,y)}{\mathrm{d}y} &= -\mathbf{F}(x,y)\frac{\mathrm{d}\mathbf{F}(y,x)}{\mathrm{d}y}\mathbf{F}(x,y) \\
&= -\mathbf{F}(x,y)\mathbf{U}(y)\mathbf{F}(y,x)\mathbf{F}(x,y) \\
&= -\mathbf{F}(x,y)\mathbf{U}(y)
\end{aligned}$$

□

1.1.1 Linear Systems on Matrix Lie Algebras.

Now, let us consider linear system (1.1.1), in the case when $\mathcal{V} = \mathbb{F}$ is a field, and we only consider $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Assume that the coefficient $\mathbf{U}(x)$ takes values in a subalgebra $\mathfrak{g}$ of $\mathbf{gl}(n,\mathbb{F})$, which is the Lie algebra consisting of $n \times n$ matrix with entries in the field $\mathbb{F}$. Denote by $\mathbf{GL}(n,\mathbb{F})$ Lie group of $n \times n$

invertible matrices. For each X a $n \times n$ real or complex matrix, the exponential of X, denoted by e^X or $\exp X$, is defined by the power series

$$\mathrm{e}^X = \sum_{i=0}^{\infty} \frac{X^m}{m!}. \tag{1.1.8}$$

The map $t \mapsto \exp tX$ is a smooth curve in $\mathbf{gl}(n, \mathbb{F})$.

Let $G \subset \mathbf{GL}(n)$ be a matrix Lie subgroup. The Lie algebra of G, denoted by $\mathfrak{g}$, is the set of matrices X such that e^{tX} is in G for all real t,

$$\mathfrak{g} := \{X \in \mathbf{gl}(n, \mathbb{F}) | \exp(tX) \in G, \forall t\}. \tag{1.1.9}$$

In fact, $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{gl}(n, \mathbb{F})$.

In general, if G is a Lie group, denote by G^0 the component connected of the element identity in G. According to [6], G^0 has the following properties:

- pathwise connected,
- open and closed subset in G,
- a normal subgroup of G.

Proposition 1.1.2. *Let $\mathfrak{g}$ be a subalgebra of $\mathbf{gl}(n, \mathbb{F})$. Fix a smooth map $\mathbf{U}(x) \in \mathfrak{g}$ such satisfy (1.1.2). If $\mathbf{F}(x, y)$ is the fundamental solution of the linear system (1.1.1) on $\mathfrak{g}$, then $\mathbf{F}(x, y)$ lies on the connected component of the identity element of $\mathbf{GL}(n)$.*

Proof. The fundamental solution $\mathbf{F}(x, y)$ is a continuous curve which joints the identity matrix $\mathbf{I}$ with $\mathbf{F}(x, y)$. Since $\mathbf{GL}^0(n)$ is pathwise connected, it follows that $\mathbf{F}(x, y)$ must belongs to the connected component of the identity. □

Consider linear system (1.1.1) on the Lie algebra $\mathfrak{g}$, with initial data $\mathbf{f}\,|_{x=y} = \mathbf{f}^0$. We define the map

$$\mathbf{f}(x) \mapsto \tilde{\mathbf{f}}(x) = \mathbf{B}(x)\mathbf{f}(x), \tag{1.1.10}$$

Here $\mathbf{B}(x)$ is a C^∞ matrix map in $\mathbf{GL}(n, \mathbb{F})$. Hence, $\tilde{\mathbf{f}}(x)$ is solution of the linear system

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{f}}(x) = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}}(x), \tag{1.1.11}$$

$$\widetilde{\mathbf{f}}(x)\,|_{x=y} = \tilde{\mathbf{f}}^0. \tag{1.1.12}$$

Here

$$\widetilde{\mathbf{U}} = \frac{\partial \mathbf{B}}{\partial x}\mathbf{B^{-1}} + \mathbf{BUB^{-1}}.$$

Proposition 1.1.3. *Let $\mathbf{F}(x, y)$ and $\widetilde{\mathbf{F}}(x, y)$ be the fundamental solutions of the linear system (1.1.11)-(1.1.12) and system (1.1.1) on $\mathfrak{g}$, respectively. Then*

$$\tilde{\mathbf{F}}(x, y) = \mathbf{B}(x)\mathbf{F}(x, y)\mathbf{B}^{-1}(x). \tag{1.1.13}$$

Proof. Since $\widetilde{\mathbf{F}}(x,y)$ is the fundamental solution of (1.1.11), we have

$$\begin{aligned}\widetilde{\mathbf{F}}(x,y)\widetilde{\mathbf{f}}^0 &= \widetilde{\mathbf{f}}(x)\\ &= \mathbf{B}(x)\mathbf{f}(x)\\ &= \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{f}^0\\ &= \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{B}^{-1}(x)\widetilde{\mathbf{f}}^0.\end{aligned}$$

Therefore,

$$\tilde{\mathbf{F}}(x,y) = \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{B}^{-1}(x).$$

□

1.1.2 Lax Equation.

We shall consider another special type of linear system, in the case when $\mathcal{V} = \mathfrak{g}$ and the coefficients $\mathbf{U}(x)$ takes values in the adjoint algebra of $\mathfrak{g}$. First we recall the definition of adjoint group and adjoint algebra.

Let G be a Lie subgroup, and let $\mathfrak{g}$ be the Lie algebra of G. The **adjoint representation** of G on $\mathfrak{g}$ is a homomorphism

$$\mathrm{Ad} \quad : \quad G \to \mathbf{GL}(\mathfrak{g}) \tag{1.1.14}$$

$$\mathrm{Ad}_g(X) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} g\exp(tX)\, g^{-1}|_{t=0}. \tag{1.1.15}$$

In particular, if G is a matrix group, then

$$\mathbf{Ad}_X(Y) := XYX^{-1}. \tag{1.1.16}$$

The **adjoint operator** of $\mathfrak{g}$ on $\mathfrak{g}$ is the homomorphism given by the differential of the adjoint representation (1.1.15).

$$\mathrm{ad} \quad : \quad \mathfrak{g} \to \mathbf{End}(n) \tag{1.1.17}$$

$$\mathrm{ad} \stackrel{\text{def}}{=} \mathrm{d}\,(\mathrm{Ad})_e\,. \tag{1.1.18}$$

thus, if $\mathfrak{g}$ is a matrix algebra, the adjoint operator is given by the matrix conmutators,

$$\mathrm{ad}_X(Y) \stackrel{\text{def}}{=} [X,Y] = XY - YX. \tag{1.1.19}$$

Let us define the **adjoint group** $\mathrm{Ad}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ as the subgroup of $\mathbf{GL}(\mathfrak{g})$ generated by the element $\mathrm{e}^{(\mathrm{Ad}X)}$, for $X \in \mathfrak{g}$. The **adjoint algebra** $\mathrm{ad}(\mathfrak{g})$ is the Lie algebra of the adjoint group.

Proposition 1.1.4. *Let $\mathfrak{g}$ be a subalgebra of $\mathbf{gl}(n,\mathbb{F})$. Fix a smooth map $\mathbf{U}(x) \in \mathrm{ad}(\mathfrak{g})$ which is bounded on $\mathbb{R}$. Then the fundamental solution $\mathbf{F}(x,y)$ of the corresponding linear system (*1.1.1*) takes values on the adjoint group,*

$$\mathbf{F}(x,y) \in \mathrm{Ad}(\mathfrak{g}) \quad \forall\ x,y. \tag{1.1.20}$$

Proof. By Proposition (1.1.2) $\mathbf{F}(x,y)$ belongs to the connected component of a Lie group whose Lie algebras are $\mathrm{ad}(\mathfrak{g})$. This connected component is $\mathrm{Ad}(G^0)$, where G^0 is the connected component of G. According to [6] we have the following facts: the adjoint group of g is the unique connected Lie subgroup of $\mathbf{GL}(\mathfrak{g})$ with Lie algebra equal to $\mathrm{ad}(\mathfrak{g})$, and hence

$$\mathrm{Ad}(G^0) = \mathrm{Ad}(\mathfrak{g}). \tag{1.1.21}$$

Therefore (1.1.21) holds $\mathbf{F}(x,y)$ is in the adjoint group. □

Under the hypothesis of Proposition (1.1.4), the data $\mathbf{U}$ and $\mathbf{f}$ in (1.1.1) can be represented as follows

$$\mathbf{U}(x) = \mathrm{Ad}_{\mathbf{A}(x)} \quad \mathbf{f}(x) = \mathbf{L}(x); \tag{1.1.22}$$

where $\mathbf{A}(x) \in \mathfrak{g}$ and $\mathbf{L}(x) \in \mathfrak{g}$. Therefore, system (1.1.1) takes the form

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}x} = [\mathbf{L}, \mathbf{A}], \tag{1.1.23}$$

This system is called the **Lax equation** [8]. In this case, condition (1.1.2) reads

$$\|\mathbf{A}(x)\| < \infty. \tag{1.1.24}$$

Proposition 1.1.5. *Under condition (1.1.24), the solution $\mathbf{L}(x)$ of Lax equation with initial data $\mathbf{L}(0) = \mathbf{L}_0$ is well defined and given by*

$$\mathbf{L}(x) = \mathrm{Ad}_{\Phi(x)}\mathbf{L}_0 = \Phi(x)\mathbf{L}_0\Phi^{-1}(x). \tag{1.1.25}$$

Where Φ is the fundamental solution of the linear system associted wiht $\mathbf{A}$,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = -\mathbf{A}\Phi, \tag{1.1.26}$$
$$\Phi(0) = \mathbf{I}. \tag{1.1.27}$$

Proof. Since $\|\mathbf{A}(x)\| < \infty$, there exists the fundamental solution $\Phi(x)$ in (1.1.26)-(1.1.27). Define

$$\widetilde{\mathbf{L}}(x) = \Phi(x)\mathbf{L}_0\Phi^{-1}(x). \tag{1.1.28}$$

Differentiating this function gives

$$\frac{\mathrm{d}\widetilde{\mathbf{L}}}{dx} = \frac{\mathrm{d}\Phi}{dx}\mathbf{L}_0\Phi^{-1} - \Phi\mathbf{L}_0\Phi^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\Phi^{-1} \tag{1.1.29}$$
$$= \Phi\mathbf{L}_0\Phi^{-1}\mathbf{A} - \mathbf{A}\Phi\mathbf{L}_0\Phi^{-1} \tag{1.1.30}$$
$$= \left[\widetilde{\mathbf{L}}, \mathbf{A}\right], \tag{1.1.31}$$

moreover, $\widetilde{\mathbf{L}}(0) = \mathbf{L}_0$. Therefore $\widetilde{\mathbf{L}}$ satisfy the Lax equation (1.1.23); and hence, by the uniqueness property, we have $\mathbf{L}(x) = \widetilde{\mathbf{L}}(x)$. □

Corollary 1.1.6. *The eigenvalues of $\mathbf{L}(x)$, $\mathrm{tr}\,\mathbf{L}^k(x)$ and $\det\mathbf{L}^k(x)$ do not depend on x, that is, they are first integral of the Lax equation.*

Proof. By Proposition (1.1.5) we have that for all x $\mathbf{L}(x)$ is conjugate to $\mathbf{L}_0$. Hence the eingenvalues of $\mathbf{L}(x)$ are the same to eigenvalues of $\mathbf{L}_0$ [5]. The rest of the corollary follows from basic properties of the trace and determinant:

$$\operatorname{tr}(\mathbf{L}(x)) = \operatorname{tr}(\Phi(x)\mathbf{L}_0\Phi^{-1}(x)) = \operatorname{tr}(\Phi(x)\Phi^{-1}(x)\mathbf{L}_0) = \operatorname{tr}(\mathbf{L}_0), \tag{1.1.32}$$

and

$$\det(\mathbf{L}(x)) = \det(\Phi(x)\mathbf{L}_0\Phi^{-1}(x)) = \det(\Phi(x)\Phi^{-1}(x)\mathbf{L}_0) = \det(\mathbf{L}_0). \tag{1.1.33}$$

Thus, they do not depend on x, only of the initial condition. □

Recall also the geometric property of the Lax equation. Denote by

$$O_V \overset{\text{def}}{=} \{\operatorname{Ad}_g(V) | g \in G\} = \operatorname{Ad}(G)V \tag{1.1.34}$$

.the adjoint orbit of $V \in \mathfrak{g}$

Corollary 1.1.7. *If L is a solution of the Lax equation (1.1.23), then $L(x) \in O_{L_0}$ for all x.*

Proof. It follows directly from the Proposition (1.1.5) and the definition of O_{L_0} given above. □

1.1.3 Zero curvature equation

Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. Let $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to \mathfrak{g}$. The equation

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}], \tag{1.1.35}$$

is called the **zero curvature equation**.

The name of zero curvature provide of and interpretation on connection theory. Let $\alpha = \mathbf{U}\mathrm{d}x + \mathbf{V}\mathrm{d}t$ be an 1-form in $\mathbb{R}^2$ with values in $\mathfrak{g}$. The zero curvature equation (1.1.35) is equivalent to

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. \tag{1.1.36}$$

This last equation is called **Maurer-Cartan**. If we interpret α as a connection in the G-bundle $\mathbb{R}^2 \times G$, then the Maurer-Cartan equation says that the curvature of this connection is zero (see [8]).

Proposition 1.1.8. *Let G be a Lie group. Let $\alpha = \mathbf{U}\mathrm{d}x + \mathbf{V}\mathrm{d}t$ be an 1-form in $\mathbb{R}^2$, where $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to \mathfrak{g}$. The following statements are equivalents:*

1. *$\mathbf{U}$ and $\mathbf{V}$ satisfy the zero curvature equation (1.1.35).*
2. *There exist a function $\mathcal{F} : \mathbb{R}^2 \to G$ such that $\alpha = \mathcal{F}^{-1}\mathrm{d}\mathcal{F}$.*

The proof of this result is out of the scope of this text, the reader interested can be found the proof in [8].

There is a closely relationship between Lax equations and zero curvature equation. Let $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to \mathfrak{g}$ be some smooth functions. Consider the following couple of Lax equations

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}x} = [\mathbf{L}, \mathbf{U}], \tag{1.1.37}$$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = [\mathbf{L}, \mathbf{V}]. \tag{1.1.38}$$

The compatibility condition for the existence of $\mathbf{L}(x,t)$, with $\mathbf{L}(0,0) = \mathbf{L}_0$, leads to the following fact: $\mathbf{U}$ and $\mathbf{V}$ satisfy the zero curvature equation (1.1.35). Moreover, there also exist a relationship between the solution $\mathbf{L}(x,t)$, $\mathbf{L}(0,0) = \mathbf{L}_0$ and the map $\mathcal{F}$ in Proposition (1.1.8), which is

$$\mathbf{L}(x,t) = \mathrm{Ad}_{\mathcal{F}^{-1}(x,t)}\mathbf{L}_0. \tag{1.1.39}$$

1.1.4 Lyapunov Transformations.

Given a change of coordinates $x \mapsto \mathbf{T}(x)$, it is possible to transform linear system (1.1.1) into other linear system. Here we shall state an equivalence relation between linear system, under a particular change of coordinates known as Lyapunov transformation.

Definition 1.1.1. *The change of coordinates*

$$\widetilde{\mathbf{f}}(x) = \mathbf{T}(x)\mathbf{f}(x), \tag{1.1.40}$$

is called a Lyapunov transformation if the C^∞ function $x \mapsto \mathbf{T}(x) \in G^0$ satisfies the following conditions:

(i) $\mathbf{T}(x)$ and $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}$ are bounded,

$$\sup_{x\in\mathbb{R}} \|\mathbf{T}(x)\| < \infty \quad \textit{and} \quad \sup_{x\in\mathbb{R}} \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x) \right\| < \infty; \tag{1.1.41}$$

(ii) There exist a real number m such that $|\det \mathbf{T}(x)| \geq m > 0$ for all $x \in \mathbb{R}$.

Remark that the inverse of a Lyapunov transformation is also a Lyapunov transformation. Indeed, we have

$$|\det \mathbf{T}(x)| \leq M < \infty. \tag{1.1.42}$$

It follows that

$$|\det \mathbf{T}^{-1}(x)| = \frac{1}{|\det \mathbf{T}(x)|} \geq \frac{1}{M}. \tag{1.1.43}$$

Using

$$\mathbf{T}^{-1}(x) = \frac{1}{|\det \mathbf{T}(x)|}\left[\Delta_{ij}\right], \tag{1.1.44}$$

by (i), (ii) we get that

$$\sup_{x\in\mathbb{R}} \|\mathbf{T}^{-1}(x)\| < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}} \left\| \frac{\mathrm{d}\mathbf{T}^{-1}}{\mathrm{d}x}(x) \right\| < \infty. \tag{1.1.45}$$

Moreover, it is easy to see that composition for two Lyapunov transformation is also Lyapunov.

Definition 1.1.2. *Two linear systems*

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f} \quad \mathbf{U}(x) \in \mathfrak{g}, \ \|\mathbf{U}(x)\| < \infty, \tag{1.1.46}$$

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}} \quad \widetilde{\mathbf{U}}(x) \in \mathfrak{g}, \ \|\widetilde{\mathbf{U}}(x)\| < \infty. \tag{1.1.47}$$

are equivalent (in the sense of Lyapunov) if they are related by a Lyapunov transformation.

Thus, the Lyapunov transformation gives an equivalence relation in the set of linear system of type (1.2.24).

Definition 1.1.3. *A linear system*

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f}, \quad \mathbf{U}(x) \in \mathfrak{g} \tag{1.1.48}$$

is called ***reducible*** *if it is equivalent to a system with constant coefficients*

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \mathbf{K}\widetilde{\mathbf{f}}, \quad \mathbf{K} \in \mathfrak{g}. \tag{1.1.49}$$

Lyapunov's Criterion. The linear system (1.2.24) is reducible if and only if its fundamental solution $\mathbf{F}(x, y)$ has a representation

$$\mathbf{F}(x, y) = \mathbf{T}(x)\mathrm{e}^{x\mathbf{K}(y)}; \tag{1.1.50}$$

where $\mathbf{T}$ is a Lyapunov transformation and $\mathbf{K}(y) \in \mathfrak{g}$ does not depend on x.

1.2 Definition of monodromy for linear boundary problems

Now we define the *monodromy matrix* and state its main properties. Let G be a subgroup of $\mathbf{GL}(n, \mathbb{F})$ and $\mathfrak{g}$ its Lie algebra. We let consider the linear system

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x, y) = \mathbf{U}(x)\mathbf{F}(x, y), \tag{1.2.1}$$

$$\mathbf{F}(x, y)|_{x=y} = \mathbf{I}. \tag{1.2.2}$$

We shall supplement the initial value problem with the following boundary condition: *periodic, quasi-periodic and decreasing case.*

1.2.1 Periodic case

We recall that the linear system (1.2.1) is periodic if there exists a real number $L > 0$ such that the matrix coefficients satisfy $\mathbf{U}(x + 2L) = \mathbf{U}(x)$ for each $x \in \mathbb{R}$. We will define the monodromy matrix in relation with the fundamental solution of the system (1.2.1). Although the material exposed here is well known [11, 12] , this case works as a model to define the monodromy matrix in the quasiperiodic and decreasing cases. The main result for periodic coefficient is the **Floquet Theorem**, which states that by means of a change of variables one can reduce equation (1.2.1) to an equation with constant coefficients.

Definition 1.2.1. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the periodic linear problem (*1.2.1*). The matrix*

$$\mathbf{M}(y) \stackrel{\mathrm{def}}{=} \mathbf{F}(y + 2L, y) \tag{1.2.3}$$

is called the ***monodromy matrix****.*

By Proposition (1.1.1), we have that $\mathbf{F}(x, y) \in G^0$ for each $x \in \mathbb{R}$ and hence $\mathbf{M}(y) \in G^0$.

Proposition 1.2.1. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the periodic linear problem (*1.2.1*).*

1. *The function $\mathbf{F}(x + 2L, y)$ is solution of the equation (*1.2.1*).*

2. *The fundamental solution* $\mathbf{F}(x,y)$ *of (1.2.1) and the map* $\mathbf{F}(x+2L,y)$ *are related by the equation*

$$\mathbf{F}(x+2L,y) = \mathbf{F}(x,y)\mathbf{M}(y), \tag{1.2.4}$$

where $\mathbf{M}(y)$ *is the monodromy matrix.*

Proof. 1. Since $\mathbf{F}(x,y)$ is fundamental solutions, by (1.2.1) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x+2L,y) = \mathbf{U}(x+2L)\mathbf{F}(x+2L,y).$$

So, $\mathbf{U}(x)$ is $2L$-periodic and

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x+2L,y) = \mathbf{U}(x)\mathbf{F}(x+2L,y).$$

Hence $\mathbf{F}(x+2L,y)$ is also a solution of (1.2.1).

2. By (1) we have the function $\mathbf{F}(x,y)$ and $\mathbf{F}(x+2L,y)$ are both solution of the same differential equation. From basic facts about theory of ODE [11],[12]; it follows that $\mathbf{F}(x,y)$ and $\mathbf{F}(x+2L,y)$ are linearly dependent. This implies the existence of a constant matrix $\mathbf{C}$, such that

$$\mathbf{F}(x+2L,y) = \mathbf{F}(x,y)\mathbf{C}. \tag{1.2.5}$$

Putting $x=y$, we obtain

$$\mathbf{C} = \mathbf{F}(2L+y,y) = \mathbf{M}(y)$$

□

Now we recall the Floquet theorem in the case when $\mathbb{F}=\mathbb{C}$. According to [6], we need the following fact [6].

Proposition 1.2.2. *Let* $G \subset \mathbf{GL}(n,\mathbb{C})$ *be a subgroup and* $\mathbf{g} \subset \mathbf{gl}(n,\mathbb{C})$ *its Lie algebra. Then*

$$\exp(\mathfrak{g}) = G^0. \tag{1.2.6}$$

Since $\mathbf{M}(y) \in G^0$, from Proposition (1.2.2), there exist a matrix $\mathbf{K} \in \mathfrak{g}$ such that

$$\mathbf{K} = \frac{1}{2L}\ln \mathbf{M}. \tag{1.2.7}$$

Theorem 1.2.3 (Floquet-Lyapunov)**.** *Let* $\mathbf{F}(x,y)$ *be fundamental solution of the linear problem (1.2.1),(1.2.2), with periodic coefficients in* $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$*. Then, the fundamental solution can be expressed in the form*

$$\mathbf{F}(x,y) = \Psi(x,y)\mathrm{e}^{x\mathbf{K}(\mathbf{y})}, \tag{1.2.8}$$

where $\mathbf{K}$ *is given by (1.2.7) and* Ψ *is a matrix function with the following properties:*

1. $\Psi(x+2L,y) = \Psi(x,y)$,

2. $\Psi(x,y) \in G^0$,

for all x.

Proof. Let us define

$$\Psi(x,y) \stackrel{\text{def}}{=} \mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)}. \tag{1.2.9}$$

It is clear that $\Psi(x,y)$ is in G^0 because is the product of $\mathbf{F}(x,y)$ with $\mathrm{e}^{-x\mathbf{K}(y)}$ both belong to G^0. We have to check that Ψ is periodic,

$$\begin{aligned}
\Psi(x+2L,y) &= \mathbf{F}(x+2L,y)\mathrm{e}^{-(x+2L)\mathbf{K}(y)} \\
&= \mathbf{F}(x,y)\mathbf{M}(y)\mathrm{e}^{-2L\mathbf{K}(y)}\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \Psi(x,y).
\end{aligned}$$

□

The classical Floquet-Lyapunov reducibility theorem provide a periodic change of variable, which reduces equation (1.2.1) to a system with constant coefficient.As a consecuence of Theorem (1.2.3) we get the following analogue of Floquet-Lyapunov theorem formulated in terms of the fundamental solution.

Theorem 1.2.4 (Lyapunov reducibility)**.** *Let* $\mathbf{K}(y)$ *given by (*1.2.7*) and* $\Psi(x,y)$ *by (*1.2.8*). Under the* $2L$*-periodic change of variables*

$$\widetilde{\mathbf{F}}(x,y) = \Psi^{-1}(x,y)\mathbf{F}(x,y)\Psi(y,y) \tag{1.2.10}$$

*the system (*1.2.1*) reduces to the form*

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{F}}(x,y) = \mathbf{K}(y)\widetilde{\mathbf{F}}(x,y), \tag{1.2.11}$$

$$\widetilde{\mathbf{F}}(x,y)|_{x=y} = \mathbf{I}. \tag{1.2.12}$$

Proof. We only must differentiate (1.2.11)

$$\frac{\mathrm{d}\widetilde{\mathbf{F}}}{\mathrm{d}x}(x,y) = -\Psi^{-1}(x,y)\frac{\mathrm{d}\Psi}{\mathrm{d}x}(x,y)\Psi^{-1}(x,y)\mathbf{F}\Psi(y,y) + \Psi^{-1}(x,y)\mathbf{U}(x)\mathbf{F}\Psi(y,y)$$

On other hand, differentiating the equation (1.2.9), gives

$$\frac{\mathrm{d}\Psi}{\mathrm{d}x}(x,y) = \mathbf{U}(x)\Psi(x,y) - \Psi(x,y)\mathbf{K}(y).$$

Substituiting the last equation into one above, we get

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\mathbf{F}}}{\mathrm{d}x} &= -\ \Psi^{-1}(x,y)\mathbf{U}(x)\Psi(x,y)\widetilde{\mathbf{F}} + \mathbf{K}(y)\widetilde{\mathbf{F}} + \Psi^{-1}(x,y)\mathbf{U}(x)\Psi(x,y)\widetilde{\mathbf{F}} \\
\frac{\mathrm{d}\widetilde{\mathbf{F}}}{\mathrm{d}x} &= \mathbf{K}(y)\widetilde{\mathbf{F}}.
\end{aligned}$$

Now, putting $x = y$ into (1.2.10) leads to

$$\widetilde{\mathbf{F}}(y,y) = \Psi^{-1}(y,y)\mathbf{F}(y,y)\Psi(y,y) = \Psi^{-1}(y,y)\Psi(y,y) = \mathbf{I}. \tag{1.2.13}$$

□

If $\mathbf{U}(x) \in \mathbf{gl}(n, \mathbb{R})$ then $\mathbf{F}(x, y)$ belongs to $\mathbf{GL}(n, \mathbb{R})$, but in general the matrices $\Psi(x, y)$ and $\mathbf{K}$ in (1.2.8) take values in a complex Lie group and complex Lie algebra, respectively. In this case, the Floquet-Lyapunov Theorem (1.2.3) is still true under the following assumption.

Corollary 1.2.5. *If the matrix coefficient in Theorem (1.2.3) takes values in $\mathfrak{g} \subset \mathbf{gl}(n, \mathbb{R})$ then fundamental solution has the form (1.2.8) only if*

$$\mathbf{M}(y) \in \exp(\mathfrak{g}) \tag{1.2.14}$$

Proof. The condition (1.2.14) guarantees that $\mathbf{K} \in \mathbf{gl}(n, \mathbb{R})$, this proves the statement. □

1.2.2 Quasiperiodic case

Let $G \subset \mathbf{GL}(n, \mathbb{F})$ be a Lie subgroup and $\mathfrak{g}$ its Lie algebra. The linear system (1.2.1) is called **quasiperiodic** if there exist a real number $L > 0$ and a constant matrix $\mathbf{Q} \in G^0$ such that the matrix coefficient $\mathbf{U}$ satisifes the condition

$$\mathbf{U}(x + 2L) = \mathbf{Q}^{-1}\mathbf{U}(x)\mathbf{Q} \tag{1.2.15}$$

for all $x \in \mathbb{R}$.

Definition 1.2.2. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the quasiperiodic linear problem (1.2.1). The matrix*

$$\mathbf{M}(y) \stackrel{\text{def}}{=} \mathbf{F}(y + 2L, y) \tag{1.2.16}$$

*is called the **monodromy**.*

In the quasiperiodic case, the monodromy $\mathbf{M}(x)$ has some additional properties.

Proposition 1.2.6. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the quasiperiodic linear problem (1.2.1) and $\mathbf{M}(y)$ its monodromy matrix. We have the following relations:*

a)

$$\mathbf{F}(x + 2L, y) = \mathbf{Q}^{-1}\mathbf{F}(x, y)\mathbf{Q}\mathbf{M}(y), \tag{1.2.17}$$

b)

$$\mathbf{F}(x + 2L, y + 2L) = \mathbf{Q}^{-1}\mathbf{F}(x, y)\mathbf{Q}, \tag{1.2.18}$$

c)

$$\mathbf{Q}\mathbf{M}(x) = \mathbf{F}^{-1}(0, x)\mathbf{Q}\mathbf{M}(0)\mathbf{F}(0, x). \tag{1.2.19}$$

d) The monodromy matrix satisfies the quasiperiodicity

$$\mathbf{M}(y + 2L) = \mathbf{Q}^{-1}\mathbf{M}(y)\mathbf{Q} \quad \forall y \in \mathbb{R}. \tag{1.2.20}$$

Proof. a) The matrix function $\mathbf{G}(x, y) = \mathbf{F}(x, y)\mathbf{Q}\mathbf{M}(y)$ is a solution of (1.1.3), because $\mathbf{F}(x, y)$ is its fundamental solution, whenever y is fixed. Evaluating at $x = y$, we have $\mathbf{G}(y, y) = \mathbf{Q}\mathbf{M}(y)$. On other hand, as we have established in Proposition (1.2.1), the function $\mathbf{H}(x, y) = \mathbf{Q}\mathbf{F}(x + 2L, y)$ is also a fundamental solution, which satisfies the same boundary condition, $\mathbf{H}(y, y) = \mathbf{Q}\mathbf{M}(y) = \mathbf{G}(y, y)$. Then, $\mathbf{H}(x, y) = \mathbf{G}(x, y)$.

b) Using the transition property, by Proposition (1.1.1), we have

$$\begin{aligned} \mathbf{F}(x+2L, y+2L) &= \mathbf{F}(x+2y, y)\mathbf{F}(y, y+2L), \\ &= \mathbf{F}(x+2L, y)\mathbf{M}^{-1}(y). \end{aligned}$$

It follows from (a) that

$$\mathbf{F}(x+2L, y+2L) = \mathbf{Q}^{-1}\mathbf{F}(x, y)\mathbf{Q}\mathbf{M}(y)\mathbf{M}^{-1}(y)$$

c) By the definition, $\mathbf{M}(0) = \mathbf{F}(2L, 0)$. Now, we have to prove the identity

$$\mathbf{F}(x, 0)\mathbf{Q}\mathbf{F}(2L, 0)\mathbf{F}(0, x) = \mathbf{F}(x, 0)\mathbf{Q}\mathbf{F}(2L, x).$$

By (b), we obtain

$$\begin{aligned} \mathbf{F}^{-1}(0, x)\mathbf{Q}\mathbf{M}(0)\mathbf{F}(\mathbf{0}, \mathbf{x}) &= \mathbf{Q}\mathbf{F}(x+2L, 2L)\mathbf{Q}^{-1}\mathbf{Q}\mathbf{F}(2L, x) \\ &= \mathbf{Q}\mathbf{F}(x+2L, x) \\ &= \mathbf{Q}\mathbf{M}(x). \end{aligned}$$

d) Directily from definition of monodromy (1.2.16) we have

$$\mathbf{M}(y+2L) = \mathbf{F}(y+4L, y+2L). \tag{1.2.21}$$

By part (b)

$$\begin{aligned} \mathbf{M}(y+2L) &= \mathbf{Q}^{-1}\mathbf{F}(y+2L, y)\mathbf{Q} && (1.2.22) \\ &= \mathbf{Q}^{-1}\mathbf{M}(y)\mathbf{Q} && (1.2.23) \end{aligned}$$

□

We shall give an analogue result to Floquet-Lyapunov theorem (1.2.3). Before we make some observation about the class of transformation which reduce a quasiperiodic linear problem (1.2.15) into a periodic problem. We recall the vectorial linear system (1.1.1)

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f}, \ \mathbf{f} = \mathbf{f}(x) \in \mathbb{F}^n \tag{1.2.24}$$

with initial data $\mathbf{f}(x, y) = \mathbf{f}^0$, when $x = y$. Assume still that the matrix coefficient $\mathbf{U}(x)$ is quasiperiodic. Let $\mathbf{T}(x) \in G^0$ be a smooth function such that

$$\mathbf{T}(x+2L) = \mathbf{T}(x)\mathbf{Q}. \tag{1.2.25}$$

We define the change of coordinates

$$\widetilde{\mathbf{f}}(x, y) = \mathbf{T}(x)\mathbf{f}(x, y). \tag{1.2.26}$$

Proposition 1.2.7. *Let $\mathbf{f}(x, y)$ be the solution of the quasiperiodic system (1.2.24). Let $\mathbf{T}(x) \in G^0$ be a matrix function which holds (1.2.25). Then the function $\widetilde{\mathbf{f}}(x, y)$ given by (1.2.26) is a solution of a periodic system.*

Proof. Differentiating (1.2.26), we obtain that $\widetilde{\mathbf{f}}(x,y)$ satisfy the system

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}}, \tag{1.2.27}$$

where

$$\widetilde{\mathbf{U}} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}\mathbf{T}^{-1} + \mathbf{T}\mathbf{U}\mathbf{T}^{-1}.$$

Only we must check that $\widetilde{\mathbf{U}}(x)$ is periodic,

$$\begin{aligned}
\widetilde{\mathbf{U}}(x+2L) &= \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x+2L)\mathbf{T}^{-1}(x+2L) + \mathbf{T}(x+2L)\mathbf{U}(x+2L)\mathbf{T}^{-1}(x+2L) \\
&= \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{T}^{-1}(x) + \mathbf{T}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{U}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{T}^{-1}(x) \\
&= \widetilde{\mathbf{U}}(x).
\end{aligned}$$

□

By now, let $\mathbb{F} = \mathbb{C}$, since $\mathbf{Q}$ and $\mathbf{M}(y)$ belong to G^0, which is a subgroup of $G \subset \mathbf{GL}(n,\mathbb{C})$, then by Proposition (1.2.2) there exist a matrix $\mathbf{K} \in \mathfrak{g}$ such that

$$\mathbf{K}(y) = \frac{1}{2L}\ln \mathbf{Q}\mathbf{M}(\mathbf{y}). \tag{1.2.28}$$

Proposition 1.2.8. *Let $\mathbf{F}(x,y)$ be fundamental solution of the linear problem (1.2.1),(1.2.2), with quasiperiodic coefficient $\mathbf{U}$ in $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$. Then, the fundamental solution can be expressed in the form*

$$\mathbf{F}(x,y) = \Psi(x,y)\mathrm{e}^{x\mathbf{K}(\mathbf{y})}, \tag{1.2.29}$$

where $\mathbf{K}$ is given by (1.2.28) and Ψ is a matrix function with the following properties:

1. $$\Psi(x+2L,y) = \mathbf{Q}^{-1}\Psi(x,y), \tag{1.2.30}$$

2. $$\Psi(x,y) \in G^0, \tag{1.2.31}$$

for all x.

Proof. Let define

$$\Psi(x,y) \stackrel{\text{def}}{=} \mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)}. \tag{1.2.32}$$

$\Psi(x,y)$ is well defined and clearly is in G^0 because is the product of $\mathbf{F}(x,y)$ with $\mathrm{e}^{-x\mathbf{K}(y)}$ both belong to G^0. We have to check that Ψ holds the property (1)

$$\begin{aligned}
\Psi(x+2L,y) &= \mathbf{F}(x+2L,y)\mathrm{e}^{-(x+2L)\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathbf{Q}\mathbf{M}(y)\mathrm{e}^{-2L\mathbf{K}(y)}\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\Psi(x,y).
\end{aligned}$$

□

If we take $T(x,y) = \Psi^{-1}(x,y)$, it follows by theorem above that $\mathbf{T}(x,y)$ holds the condition (1.2.25). Hence, the Proposition (1.2.7) imply that $\mathbf{T}$ produces a change of coordinates that reduces the quasiperiodic linear system (1.2.1) to a system with periodic coefficient. Even more, it is reduced to a system with constant coefficient as we are going to establish in the next result.

Theorem 1.2.9. *Let $\mathbf{K}(y)$ and $\Psi(x,y)$ be matrix valued function given by (1.2.28) and (1.2.29), respectively. Under the transformation of coordinates*

$$\widetilde{\mathbf{F}}(x,y) = \Psi^{-1}(x,y)\mathbf{F}(x,y)\Psi(y,y) \tag{1.2.33}$$

quasiperiodic system (1.2.1) is reduced to the system with constant coefficients

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{F}}(x,y) = \mathbf{K}(y)\widetilde{\mathbf{F}}(x,y), \tag{1.2.34}$$

$$\widetilde{\mathbf{F}}(x,y)|_{x=y} = \mathbf{I}. \tag{1.2.35}$$

Proof. We have to proceed in the same way as in the proof of the periodic reducibility Theorem (1.2.4). □

We can make some observations about results obtained in the Theorem (1.2.4) and Proposition (1.2.9) for periodic and quasiperiodic case , respectively; in the context of *Lyapunov reducibility.* According to Lyapunov criterion we can arrive to the following conclusions:

- In the periodic case the Theorem (1.2.3) states that the fundamental solution $\mathbf{F}(x,y)$ has the form (1.1.50)and that $\Psi(x,y)$ is periodic. This fact implies that Ψ is bounded,

  $$\sup_{x\in\mathbb{R}} \|\Psi(x,y)\| = \sup_{x\in[0,2L]} \|\Psi(x,y)\| < \infty, \tag{1.2.36}$$

  and analogously its derivative also is bounded. Hence,every periodic linear system is reducible in the sense of Lyapunov, as was stated in Theorem (1.2.11).

- In the quasiperiodic case the situation is a little different, because we have to include an additional condition for that quasiperiodic system will be reducible. By Proposition (1.2.8) we find that $\mathbf{F}(x,y)$ also can be expresed in the form (1.1.50), and Ψ satisfy

  $$\Psi(x+2L,y) = \mathbf{Q}^{-1}\Psi(x,y). \tag{1.2.37}$$

  Therefore, a quasiperiodic system is reducible if

  $$\|\mathbf{Q}\| < 1 \quad \text{or} \quad \|\mathbf{Q}^{-1}\| < 1. \tag{1.2.38}$$

1.2.3 Decreasing case

We shall analyze the properties of the fundamental solution $\mathbf{F}(x,y)$ on the whole axis $-\infty < x,y < \infty$, under the assumption that $\mathbf{U}_{ij}(x)$ vanish as $|x| \to \infty$. More precisely, these functions will be supposed absolutely integrable on $\mathbb{R}^1$, i.e. $\mathbf{U}_{ij}(x)$ lies in $\mathbf{L}_1(-\infty,\infty)$. In terms of the matrix coefficient $\mathbf{U}(\mathbf{x})$ of the system

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{F},$$

this results equivalent to

$$\int_{-\infty}^{\infty} \|\mathbf{U}(x)\| dx < \infty, \tag{1.2.39}$$

where $\|\cdot\|$ is some matricial norm. In the successive, the space of $n \times n$ matrix functions satisfaying (1.2.39) will be denoted by $\mathbf{L}_1^{n\times n}(-\infty,\infty)$.

Under this assumption we will show the next result.

Proposition 1.2.10. *There exists the limits*

$$\mathbf{F}_{\pm}(x) = \lim_{y\to\pm\infty} \mathbf{F}(x,y), \tag{1.2.40}$$

for each x in $\mathbb{R}^1$. Moreover, this limits hold the integral representation

$$\mathbf{F}_{-}(x) = \mathbf{I} + \int_{-\infty}^{x} \mathbf{U}(z)\mathbf{F}_{-}(z)dz, \tag{1.2.41}$$

$$\mathbf{F}_{+}(x) = \mathbf{I} - \int_{x}^{\infty} \mathbf{F}_{+}(z)\mathbf{U}(z)dz. \tag{1.2.42}$$

Proof. Recall that $\mathbf{F}(x,y)$ holds the integral equation

$$\mathbf{F}(x,y) = \mathbf{I} + \int_{y}^{x} \mathbf{U}(z)\mathbf{F}(z,y)dz,$$

then

$$\|\mathbf{F}(x,y)\| \leq 1 + \int_{y}^{x} \|\mathbf{U}(z)\|\|\mathbf{F}(z,y)\|dz,$$

assuming, without loss of generality $\|\mathbf{I}\| = 1$. Using the Gronwall's inequality, we have

$$\|\mathbf{F}(x,y)\| \leq \exp\int_{y}^{x} \|\mathbf{U}(z)\|dz,$$

taking limit as $y \to -\infty$, it follows

$$\|\mathbf{F}_{-}(x)\| \leq \exp\int_{-\infty}^{x} \|\mathbf{U}(z)\|dz < \infty,$$

since $\mathbf{U}(x)$ lies on $\mathbf{L}_1^{n\times n}(\mathbb{R})$; therefore $\mathbf{F}_{-}(x)$ exists for each real x. Now we prove the integral representation for $\mathbf{F}_{-}(x)$, first we have to prove that the map $\mathbf{U}(z)\mathbf{F}_{-}(z)$, as function of z, belongs to $\mathbf{L}_1(-\infty, x)$, for each x fixed. From the integral equation for the fundamental solution,

$$\mathbf{U}(x)\mathbf{F}(x,y) = \mathbf{U}(x) + \mathbf{U}(x)\int_{y}^{x} \mathbf{U}(z)\mathbf{F}(z,y)dz,$$

then

$$\|\mathbf{U}(x)\mathbf{F}(x,y)\| \leq \|\mathbf{U}(x)\| + \|\mathbf{U}(x)\|\int_{y}^{x} \|\mathbf{U}(z)\mathbf{F}(z,y)\|dz,$$

integrating over $y < s < x$, we obtain

$$\int_{y}^{x} \|\mathbf{U}(s)\mathbf{F}(s,y)\|ds \leq \int_{y}^{x} \|\mathbf{U}(s)\|ds + \int_{y}^{x} \|\mathbf{U}(s)\| \int_{y}^{s} \|\mathbf{U}(z)\mathbf{F}(z,y)\|dzds.$$

By Gronwall's inequality, it follows

$$\begin{aligned} \int_{y}^{x} \|\mathbf{U}(s)\mathbf{F}(s,y)\|ds &\leq \int_{y}^{x} \|\mathbf{U}(s)\|\mathrm{ds}\left(\exp\left\{\int_{y}^{x} \|\mathbf{U}(s)\|ds\right\}\right) \\ &\leq \int_{-\infty}^{x} \|\mathbf{U}(s)\|\mathrm{ds}\left(\exp\left\{\int_{-\infty}^{x} \|\mathbf{U}(s)\|ds\right\}\right) < \infty. \end{aligned}$$

From the last estimate, we conclude that

$$\int_{-\infty}^{x} \|\mathbf{U}(z)\mathbf{F}_{-}(z)\| \mathrm{d}z < \infty.$$

It which implies $\mathbf{U}(z)\mathbf{F}_{-}(z)$ lies on $\mathbf{L}_1(-\infty, x)$. Even more, we can take limit as $y \to -\infty$ in the integral equation for $\mathbf{F}(x, y)$ and obtain

$$\mathbf{F}_{-}(x) = \mathbf{I} + \int_{-\infty}^{x} \mathbf{U}(z)\mathbf{F}_{-}(z)dz.$$

The existence of the limit (1.3.26) as $y \to \infty$ is established by similar arguments; recalling that $\mathbf{F}^{-1}(x, y) = \mathbf{F}(y, x)$, we can see that $F_{+}^{-1}(x)$ exists and has the integral representation

$$\mathbf{F}_{+}^{-1}(x) = \mathbf{I} + \int_{x}^{\infty} \mathbf{U}(z)\mathbf{F}_{+}^{-1}(z)dz.$$

By other hand

$$\begin{aligned} \frac{d\mathbf{F}_{+}(x)}{dx} &= \mathbf{F}_{+}(x)\frac{d\mathbf{F}_{+}^{-1}(x)}{dx}\mathbf{F}_{+}(x) \\ &= \mathbf{F}_{+}(x)\mathbf{U}(x) \end{aligned}$$

the last expression is amounts to

$$\mathbf{F}_{+}(x) = \mathbf{I} - \int_{x}^{\infty} \mathbf{F}_{+}(z)\mathbf{U}(z)dz.$$

□

Corollary 1.2.11. *The maps $\mathbf{F}_{\pm}(x)$, satisfies the linear problem*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}_{\pm}(x) = \mathbf{U}(x)\mathbf{F}_{\pm}(x) \tag{1.2.43}$$

with the asymptotic conditions

$$\mathbf{F}_{\pm}(x) = \mathbf{I} \quad \textit{if} \quad x \to \pm\infty$$

Proof. Differentiating the integral equation for $\mathbf{F}_{\pm}(x)$ with respect x and taking limits as $y \to \pm\infty$ it deduces, easily, the differential equation and the asymptotic conditions. □

Now, we are ready to establish the concept of Monodromy in this case.

Definition 1.2.3. *The matrix*

$$\mathbf{M} = \mathbf{F}_{+}^{-1}(x)\mathbf{F}_{-}(x), \tag{1.2.44}$$

*is the **Monodromy matrix** in the decreasing case.*

One fact that is not evidented from the above definition, is that $\mathbf{M}$ is constant for all $-\infty < x < \infty$. Even more, it enjoys of some useful properties as we shall establish in the next proposition.

Proposition 1.2.12. *The monodromy matrix $\mathbf{M}$ has the properties*

a) $\mathbf{M}$ *is independent at x.*

b) $\mathbf{F}(x, y) = \mathbf{F}_{+}(x)\mathbf{M}\mathbf{F}_{-}^{-1}(y)$

Proof. a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{M} &= \frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_+^{-1}(x))\mathbf{F}_-(x) + \mathbf{F}_+^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_-(x)) \\ &= -\mathbf{F}_+^{-1}\mathbf{U}(x)\mathbf{F}_-(x) + \mathbf{F}_+^{-1}(x)\mathbf{U}(x)\mathbf{F}_-(x) \\ &= \mathbf{0}\end{aligned}$$

b) Considering again the fundamental solution $\mathbf{F}(x, y)$, we only need to verify the map $\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y)$ also satisfy the linear problem at the same boundary condition.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y)) = \mathbf{U}(x)\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y) \tag{1.2.45}$$

Now, if we put $x = y$ then

$$\mathbf{F}_+(y)\mathbf{M}\mathbf{F}_-^{-1}(y) = \mathbf{F}_+(y)\mathbf{F}_+^{-1}(y)\mathbf{F}_-(y)\mathbf{F}_-^{-1}(y) = \mathbf{I}$$

□

As a consecuence from the part b of the proposition above, We can deduce that the monodromy matrix can be computed taking some limits, as we shall see in the next corollary.

Corollary 1.2.13. *The monodromy matrix can be expresed as*

$$\mathbf{M} = \lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}(x, y) \tag{1.2.46}$$

Proof. By part (b) of the proposition (1.2.12) and corollary (1.2.11), it follow that

$$\begin{aligned}\lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}(x, y) &= \lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y) \\ &= \lim_{x\to\infty}\mathbf{F}_+(x)\left(\lim_{y\to-\infty}\mathbf{M}\mathbf{F}_-^{-1}(y)\right) \\ &= \lim_{x\to\infty}\mathbf{F}_+(x)\mathbf{M} \\ &= \mathbf{M}\end{aligned}$$

□

The formula (1.2.46) can be simplified if we put $x = L$ an $y = -L$

$$\mathbf{M} = \lim_{L\to\infty}\mathbf{F}(L, -L) \tag{1.2.47}$$

The function $\mathbf{F}(L, -L)$ on the right hand side of (1.2.47) coincides precisely with the monodromy matrix $\mathbf{M}(L)$ of the periodic and quasiperiodic case, therefore the form $\mathbf{F}(L, -L)$ can be played the role the monodromy matrix for the decreasing case, regarded as the infinite period limit, as $L \to \infty$, of the periodic monodromy matrix with the oscilating factors reduced out.

1.3 Analytic Properties of Fundamental solution

Let $G \subset \mathbf{GL}(n, \mathbb{C})$ be a Lie subgroup and $\mathfrak{g}$ its Lie algebra. Consider the linear system

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \left[\mathbf{U}_0(x) + \lambda \mathbf{U}_1(x) + \lambda^2 \mathbf{U}_2(x) + ...\right] \mathbf{f}, \tag{1.3.1}$$

where $\mathbf{U}_i$ are smooth functions on a compact domain $D \subset \mathbb{R}$. We assume that if $|\lambda| < r_0$, the series

$$\sum_{k=0}^{\infty} \lambda^k \int_D |\mathbf{U}_k(x)| dx \tag{1.3.2}$$

is convergent.

Proposition 1.3.1. *Let $\mathbf{F}(x, y, \lambda)$ be the fundamental solution of (1.3.1). Under the assumption (1.3.2) we have that the sequence on function*

$$\begin{aligned} \mathbf{F}_0(x, y, \lambda) &= \mathbf{I}, & (1.3.3)\\ \mathbf{F}_k(x, y, \lambda) &= \int_y^x \left[\sum_{l=0}^{\infty} \lambda^l \mathbf{U}_l(\tau)\right] \mathbf{F}_{k-1}(\tau, \lambda) \mathrm{d}\tau, \quad (l = 1, 2, ...) & (1.3.4) \end{aligned}$$

are analytic function and converge uniformly in x, λ to $\mathbf{F}(x, y, \lambda)$. More over, if the the matrix coefficients $\mathbf{U}_0(x) + \lambda \mathbf{U}_1(x) + \lambda^2 \mathbf{U}_2(x) + ...$ is an entire funtion on λ, then $\mathbf{F}(, \lambda)$ is an entire function of λ for fixed x.

We do not present the proof here, which can be seen (????)

1.3.1 Case $\mathbf{U} = \mathbf{U}_0 + \lambda \mathbf{U}_1$

Let consider the case when the matrix coefficient of system (1.3.1) takes the form

$$\mathbf{U}(x, \lambda) = \mathbf{U}_0(x) + \lambda \mathbf{U}_1, \tag{1.3.5}$$

where $\mathbf{U}_0(x) \in \mathfrak{g}$ is a smooth function and $\mathbf{U}_1 \in \mathfrak{g}$ is an invertible constant matrix. When the matrix coefficient is as (1.3.5), linear system (1.3.1) can be analized as an spectral problem

$$\mathscr{L}\mathbf{f} = \lambda \mathbf{f}, \tag{1.3.6}$$

with the differential operator

$$\mathscr{L} = \mathbf{U}_1^{-1} \frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{U}_1^{-1} \mathbf{U}_0(x). \tag{1.3.7}$$

Integral equation

Here, we derive integral equation for the fundamental solution of (1.3.1). First, we note that linear system (1.3.1) is equivalent to

$$\mathbf{F}(x, y, \lambda) = \mathbf{I} + \int_y^x \mathbf{U}(z, y, \lambda) \mathbf{F}(z, y, \lambda) \mathrm{d}z. \tag{1.3.8}$$

Let $\mathbf{E}(\mathbf{x} - \mathbf{y}, \lambda)$ be the fundamental solution of the linear system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E} &= \lambda \mathbf{U}_1 \mathbf{E}, & (1.3.9)\\ \mathbf{E}(0, \lambda) &= \mathbf{I}, \quad \text{for each } \lambda \in \mathbb{R}. & (1.3.10) \end{aligned}$$

Now, let $\mathbf{G}(x, y, \lambda)$ and $\mathbf{H}(x, y, \lambda)$ be, respectively, the fundamental solutions of the following differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{G}(x, y, \lambda) &= \mathbf{E}(y - x, \lambda)\mathbf{U}_0(x)\mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda) \\ \mathbf{G}(x, y, \lambda)|_{x=y} &= \mathbf{I}, \end{aligned} \tag{1.3.11}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\mathbf{H}(x, y, \lambda) &= -\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda)\mathbf{U}_0(y)\mathbf{E}(y - x, \lambda) \\ \mathbf{H}(x, y, \lambda)|_{x=y} &= \mathbf{I}. \end{aligned} \tag{1.3.12}$$

Lemma 1.3.2. *The fundamental solution* $\mathbf{F}(x, y, \lambda)$ *of the linear problem (*1.3.1*) can be expressed as*

$$\mathrm{F}(x, y, \lambda) = \mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda) \tag{1.3.13}$$

and

$$\mathrm{F}(x, y, \lambda) = \mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda) \tag{1.3.14}$$

Proof. This proof is straight forward calculation, we just must check that the right hand side of (1.3.13)-(1.3.14) both satisfy the same Cauchy problem that $\mathbf{F}(x, y, \lambda)$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda) &= \lambda\mathbf{U}_1\mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda) + \mathbf{U}_0(x)\mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda) \\ &= \mathbf{U}(x, \lambda)\mathbf{E}(x - y, \lambda)\mathbf{G}(x, y, \lambda). \end{aligned}$$

on other hand

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda) &= -\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda)\mathbf{U}_0(y) - \mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda)\lambda\mathbf{U}_1 \\ &= -\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda)\mathbf{U}(y, \lambda), \end{aligned}$$

this equation implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda) = \mathbf{U}(x, \lambda)\mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda). \tag{1.3.15}$$

so (1.3.13)-(1.3.14) hold. □

In the next result we established a couple of integral representation for $\mathbf{F}(x, y, \lambda)$, that will be useful in the preceding sections.

Proposition 1.3.3. *The fundamental solution* $\mathbf{F}(x, y, \lambda)$ *of de linear problem (*1.3.21*)-(*1.3.22*) has the following integral representations*

$$\mathbf{F}(x, y, \lambda) = \mathbf{E}(x - y, \lambda) + \int_y^x \mathbf{F}(x, z, \lambda)\mathbf{U}_0(z)\mathbf{E}(z - y, \lambda)\mathrm{d}z, \tag{1.3.16}$$

$$\mathbf{F}(x, y, \lambda) = \mathbf{E}(x - y, \lambda) + \int_y^x \mathbf{E}(x - z, \lambda)\mathbf{U}_0(z)\mathbf{F}(z, y, \lambda)\mathrm{d}z. \tag{1.3.17}$$

Proof. First we note the linear systems (1.3.11) and (1.3.12) are equivalent to the integral equations

$$\mathbf{G}(x,y,\lambda) = \mathbf{I} + \int_y^x \mathbf{E}(y-z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z-y,\lambda)\mathbf{G}(z,y,\lambda)\mathrm{d}z, \tag{1.3.18}$$

$$\mathbf{H}(x,y,\lambda) = \mathbf{I} + \int_y^x \mathbf{H}(x,z,\lambda)\mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z-x,\lambda)\mathrm{d}z. \tag{1.3.19}$$

We only must multiply $\mathbf{E}(x-y,\lambda)$ on the left with (1.3.18) and on the right with (1.3.19), respectively, so use the identities (1.3.13),(1.3.14). □

Recall that we are considering that the matrix coefficient $\mathbf{U}(x,\lambda)$ is defined for x on a compact domain. The Proposition (1.3.1) establish that $\mathbf{F}(x,y,\lambda)$ is analytic on λ for each x. Using the integral representation (1.3.16),(1.3.17) and the formula of integration by parts we can derive the asymptotic expansion for $\mathbf{F}(x,y,\lambda)$

$$\begin{aligned}\mathbf{F}(x,y,\lambda) \quad &= \quad \mathbf{E}(x-y,\lambda) + \sum_{n=1}^{\infty} \frac{\mathbf{F}(x,y,\lambda)}{\lambda^n}\mathbf{E}(x-y,\lambda) \\ &\quad + \sum_{n=1}^{\infty} \frac{\widetilde{\mathbf{F}}(x,y,\lambda)}{\lambda^n}\mathbf{E}(y-x,\lambda) + O(|\lambda|^{-\infty}).\end{aligned} \tag{1.3.20}$$

Without loss of generality, we can assume valid of the result exposed here, for the periodic and quasiperidic boundary condition. Because in both case we can reduce the analysis to a fundamental domain which size is given by the period. Therefore the Monodromy matrix holds, in these case, the same analytic properties of the fundamental solution.

Decreasing case

A special treatment must be done for the decreasing condition for to derive the analytic properties of the fundamental solution and monodromy. Now, we consider the linear system

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} \quad = \quad \mathbf{U}(x,\lambda), \tag{1.3.21}$$

$$\mathbf{F}(x,y,\lambda)|_{x=y} \quad = \quad \mathbf{I}, \tag{1.3.22}$$

where the matrix function $\mathbf{U}(x,\lambda)$, takes the form

$$\mathbf{U}(x,\lambda) = \mathbf{U}_0(x) + \lambda\mathbf{U}_1, \tag{1.3.23}$$

with λ is a real parameter, $\mathbf{U}_0(x) \in L_1^{n\times n}(-\infty,\infty)$ and $\mathbf{U}_1$ is a constant matrix in $\mathfrak{g}$.

Before to continue, we have to check that in a compact domain the integral equation (1.3.16),(1.3.17) still remain valid and so we shall extend it to whole real.

Lemma 1.3.4. *Let* $\mathbf{U}_0(x)$ *be a function in* $L_1^{n\times n}(-\infty,\infty)$. *Then the map* $\mathbf{E}(y-x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x-y,\lambda)$ *also belongs to* $L_1^{n\times n}(-\infty,\infty)$.

Proof. Recall that $\mathbf{E}(x-y,\lambda)$ is the fundamental solution on the linear system (1.3.9)-(1.3.10), so it has the integral representation

$$\mathbf{E}(x-y,\lambda) = \mathbf{I} + \int_y^x \lambda\mathbf{U}_1\mathbf{E}(s-y,\lambda)ds. \tag{1.3.24}$$

Then, by Gronwall's inequality [11], we have

$$\|\mathbf{E}(x-y,\lambda)\| \leq \exp\{\lambda \int_y^x \|\mathbf{U}_1\| ds\}. \tag{1.3.25}$$

Thus,

$$\begin{aligned} \|\mathbf{E}(y-x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x-y,\lambda)\| &\leq \exp\{\lambda \int_x^y \|\mathbf{U}_1\| ds\}\|\mathbf{U}_0(x)\| \exp\{\lambda \int_y^x \|\mathbf{U}_1\| ds\} \\ &\leq \exp\{-\lambda \int_y^x \|\mathbf{U}_1\| ds\} \exp\{\lambda \int_y^x \|\mathbf{U}_1\| ds\}\|\mathbf{U}_0(x)\| \\ &\leq \|\mathbf{U}_0(x)\|, \end{aligned}$$

hence, the map $\mathbf{E}(y-x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x-y,\lambda)$ is absolutely integrable on $\mathbb{R}$. □

Therefore, the integral equation (1.3.16),(1.3.17) are also valid for the rapidly decreasing condition.

Proposition 1.3.5. *Let $\mathbf{F}(x,y,\lambda)$ be the fundamental solution of the system (1.3.21)-(1.3.22).*

(i) The maps

$$\mathbf{F}_\pm(x,\lambda) = \lim_{y\to\pm\infty} \mathbf{F}(x,y,\lambda)\mathbf{E}(y,\lambda), \tag{1.3.26}$$

are well defined for each x and λ in $\mathbb{R}$.

(ii) $\mathbf{F}_\pm(x,\lambda)$ satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}_\pm = \mathbf{U}(x,\lambda)\mathbf{F}_\pm, \tag{1.3.27}$$

with the asymptotic conditions

$$\mathbf{F}_\pm(x,\lambda) \to \mathbf{E}(x,\lambda) \quad \text{as} \quad x \to \pm\infty. \tag{1.3.28}$$

Proof. (i) By the Lemma (1.3.2), the fundamental solution $\mathbf{F}(x,y,\lambda)$, can be written

$$\mathbf{F}(x,y,\lambda) = \mathbf{H}(x,y,\lambda)\mathbf{E}(x-y,\lambda).$$

Thus,

$$\mathbf{F}(x,y,\lambda)\mathbf{E}(y,\lambda) = \mathbf{H}(x,y,\lambda)\mathbf{E}(x,\lambda),$$

since the function $\mathbf{H}(x,y,\lambda)$ is the solution of a system whose coefficients are in $L_1^{n\times n}$, (1.3.12), it follows from Proposition (1.2.10) that the limits

$$\mathbf{H}_\pm(x,\lambda) = \lim_{y\to\pm\infty} \mathbf{H}(x,y,\lambda)$$

exist. Therefore

$$\mathbf{F}_\pm(x,\lambda) = \lim_{y\to\pm\infty} \mathbf{H}(x,y,\lambda)\mathbf{E}(x,\lambda) = \mathbf{H}_\pm(x,\lambda)\mathbf{E}(x,\lambda)$$

is well defined.

(ii)

□

Definition 1.3.1. *The monodromy matrix for the decreasing case with spectral parameter* λ *is defined*

$$\mathbf{M}(\lambda) = \mathbf{F}_+^{-1}(x,\lambda)\mathbf{F}_-(x,\lambda) \tag{1.3.29}$$

Proposition 1.3.6. *The monodromy matrix* $\mathbf{M}$ *has the properties:*

a) $\mathbf{M}(\lambda)$ *is independent of* x.

b) $\mathbf{F}(x,y,\lambda) = \mathbf{F}_+(x,\lambda)\mathbf{M}(\lambda)\mathbf{F}_-^{-1}(y,\lambda)$

1.4 The Time Evolution of the Monodromy Matrix.

Let $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$ be a Lie subalgebra. We shall assume that the matrix functions

$$(x,t) \mapsto \mathbf{U}(x,t) \in \mathfrak{g},$$
$$(x,t) \mapsto \mathbf{V}(x,t) \in \mathfrak{g},$$

are given and satisfy the **Zero Curvature Equation**

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0. \tag{1.4.1}$$

For every fixed t that $\mathbf{F}(x,y,t)$ is the fundamental solution of the linear problem associated with $\mathbf{U}(x,t)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x,y,t) = \mathbf{U}(x,t)\mathbf{F}(x,y,t), \tag{1.4.2}$$
$$\mathbf{F}(x,y,t)|_{x=y} = \mathbf{I}. \tag{1.4.3}$$

We have the following basic lemma.

Lemma 1.4.1. *If* $\mathbf{F}(x,y,t)$ *is the solution of the system (*1.4.2*),(*1.4.3*), then*

$$\frac{\partial}{\partial t}\mathbf{F}(x,y,t) = \mathbf{V}(x,t)\mathbf{F}(x,y,t) - \mathbf{F}(x,y,t)\mathbf{V}(y,t) \tag{1.4.4}$$

Proof. By straigth forward computation we derive

$$\begin{aligned}\frac{\partial^2 \mathbf{F}}{\partial x \partial t} &= \frac{\partial \mathbf{U}}{\partial t}\mathbf{F} + \mathbf{U}\frac{\partial \mathbf{F}}{\partial t}, \\ &= \frac{\partial \mathbf{V}}{\partial x}\mathbf{F} + \mathbf{VUF} - \mathbf{UVF} + \mathbf{U}\frac{\partial \mathbf{F}}{\partial t}.\end{aligned}$$

Here $\mathbf{U}$ and $\mathbf{V}$ satisfy the zero curvature equation. Next,

$$\frac{\partial^2 \mathbf{F}}{\partial x \partial t} = \frac{\partial}{\partial x}(\mathbf{VF}) + \mathbf{U}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}),$$

and

$$\frac{\partial}{\partial x}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}) = \mathbf{U}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}).$$

So, $\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}$ is also a solution of equation (1.4.2). This implies the existence of a non-singular matrix $\mathbf{C}$, independent of x, such that

$$\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF} = \mathbf{FC},$$

putting $x = y$, we obtain

$$\mathbf{C}(y,t) = -\mathbf{V}(y,t)$$

□

Now, let us think of the variable t as time and study the evolution of the monodromy matrix.

Proposition 1.4.2. *Assume that the matrix maps* $\mathbf{U}(x,t)$ *and* $\mathbf{V}(x,t)$ *satisfy de zero curvature equation.*

i) Let $\mathbf{U}(x,t)$ *and* $\mathbf{V}(x,t)$ *be quasi-periodic, i.e.,*

$$\mathbf{U}(x+2L,t) = \mathbf{Q}^{-1}\mathbf{U}(x,t)\mathbf{Q},$$
$$\mathbf{V}(x+2L,t) = \mathbf{Q}^{-1}\mathbf{V}(x,t)\mathbf{Q},$$

then the monodromy matrix $\mathbf{M}(y,t)$ *satisfies the Lax equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{MQ}) = [\mathbf{V}(L,t), \mathbf{MQ}]. \tag{1.4.5}$$

ii) If for every fixed t*, the map* $\mathbf{U}(x,t)$ *lies in* $\mathbf{L}_1^{n\times n}(-\infty,\infty)$ *and*

$$\lim_{x\to\pm\infty} \mathbf{V}(x,t) = \mathbf{V}_0,$$

where $\mathbf{V}_0$ *is a constant matrix, then*

$$\frac{d\mathbf{M}}{dt} = [\mathbf{V}_0, \mathbf{M}]. \tag{1.4.6}$$

Proof. From the basic lemma (1.4.1), putting $x = L$ and $y = -L$

$$\frac{\partial}{\partial t}\mathbf{F}(L,-L,t) = \mathbf{V}(L,t)\mathbf{F}(L,-L,t) - \mathbf{F}(L,-L,t)\mathbf{V}(-L,t), \tag{1.4.7}$$

now, we prove each incise,

i) Recall that in the quasiperiodic case the monodromy matrix

$$\mathbf{M}_L(t) = \mathbf{F}(L,-L,t)$$

multiplying the equation (1.4.7) by $\mathbf{Q}$ and using the identity $\mathbf{QV}(L,t) = \mathbf{V}(-L,t)\mathbf{Q}$, we obtain the result desired.

ii) For this case, we have been proved before

$$\mathbf{M}(t) = \lim_{L\to\infty} \mathbf{F}(L,-L,t).$$

only we must take limit as $L \to \infty$ to equation (1.4.7)

□

As direct consequence from the proposition above we have an useful characteristic of the monodromy matrix. Before, we recall an basic important result, which is known as **Liouville's Formula**

Lemma 1.4.3. *Let* $\mathbf{A}(t)$ *be a smooth real valued matrix function in* $\mathrm{GL}(n,\mathbb{C})$*. The derivative of the determinant* $\det \mathbf{A}(t)$ *is given by the formula*

$$\frac{1}{\det \mathbf{A}(t)} \frac{\mathrm{d}}{\mathrm{dt}} \det \mathbf{A}(t) = \mathrm{tr}\left(\mathbf{A}^{-1}(t)\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}(t)\right). \tag{1.4.8}$$

Theorem 1.4.4. *Let* $\mathbf{F}(x,y)$ *be the fundamental solution of (*1.1.1*), and let* $\mathbf{M}(t)$ *be the monodromy matrix given by:*

- $\mathbf{M}(y) = \mathbf{F}(2L+y,y)$*, in quasiperiodic case, and*
- $\mathbf{M}(\lambda) = \mathbf{F}_+^{-1}(x,\lambda)\mathbf{F}_-(x,\lambda)$ *in the decreasing case.*

In both cases

$$\mathrm{tr}(\mathbf{M}(t)) \quad \textit{and} \quad \det(\mathbf{M}(t))$$

are independent of the variable t*.*

Proof. An important fact, essential in this proof, is that the trace of a square matrix and the derivative are two commutative linear operator, i.e., for any $\mathbf{A}(t)$ smooth matrix, we have

$$\mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{tr}\,\mathbf{A}(t)\right).$$

In the decreasing case the Proposition 1.4.2 acclaim the monodromy matrix $\mathbf{M}(t)$ satisfy the Lax equation (1.4.6), if we calculate the trace in both sides of that equation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{tr}\,\mathbf{M}(t)\right) &= \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(t)\right)\\ &= \mathrm{tr}[\mathbf{V}_0,\mathbf{M}] = 0.\end{aligned}$$

The Lax equation (1.4.5) for quasiperiodic case imply

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{M}) = \mathbf{VM} - \mathbf{MQVQ}^{-1}. \tag{1.4.9}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{tr}\,\mathbf{M}(t)\right) &= \mathrm{tr}\left(\mathbf{VM} - \mathbf{MQVQ}^{-1}\right)\\ &= \mathrm{tr}\left(\mathbf{VM}\right) - \mathrm{tr}\left(\mathbf{MQVQ}^{-1}\right)\\ &= \mathrm{tr}\left(\mathbf{VM}\right) - \mathrm{tr}\left(\mathbf{VM}\right)\\ &= 0.\end{aligned}$$

Since in any case the derivative of the trace is equal to zero it follow that the trace of monodromy matrix is time independent.

Now, we let arrive at the same conclusion for the determinant of monodromy matrix. From the equation (1.4.9), for quasiperiodic case, and the Liouville formula (1.4.8), we have

$$\begin{aligned}
\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbf{M}) &= \operatorname{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{M}(t)\right)\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V}\mathbf{M} - \mathbf{Q}\mathbf{V}\mathbf{Q}^{-1}\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V}\mathbf{M}\right) - \operatorname{tr}\left(\mathbf{Q}\mathbf{V}\mathbf{Q}^{-1}\right) \\
&= 0.
\end{aligned}$$

On other hand, in the decreasing case, it follows from Lax equation (1.4.6) and Liouville formula again (1.4.8)

$$\begin{aligned}
\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbf{M}) &= \operatorname{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{M}(t)\right)\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V_0}\mathbf{M} - \mathbf{V_0}\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V_0}\mathbf{M}\right) - \operatorname{tr}\left(\mathbf{V_0}\right) \\
&= 0.
\end{aligned}$$

As we can see, in both case we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbf{M}) = 0,$$

therefore, $\det \mathbf{M}$ is not depending on variable t. □

Chapter 2

Linear problem in $\mathfrak{sl}(2,\mathbb{C})$

In this Chapter we apply the general results obtained in Chapter 1 to the study of linear problems on the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ possessing the involution property. Such a class of linear problems plays an important role in the integrability theory for nonlinear evolution equations.

2.1 The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

Let us consider the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \subset \mathbf{gl}(2,\mathbb{C})$ consisting of all 2×2 matrices with zero trace,

$$\mathfrak{sl}(2,\mathbb{C}) = \{\mathbf{A} \in \mathbf{gl}(2,\mathbb{C}) | \operatorname{tr} \mathbf{A} = 0\}\,. \tag{2.1.1}$$

Each element of $\mathfrak{sl}(2,\mathbb{C})$ can be written as a linear combination in some basis of $\mathfrak{sl}(2,\mathbb{C})$ [5]. The Pauli matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1.2}$$

form a basis of $\mathfrak{sl}(2,\mathbb{C})$. Consider also the matrices

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.1.3}$$

It is clear that $\{\sigma_+, \sigma_-, \sigma_3\}$ is also a base of $\mathfrak{sl}(2,\mathbb{C})$.

We need the following properties of Pauli matrices:

(a) Idempotent property:

$$\sigma_i \cdot \sigma_i = \mathbf{I}, \quad \text{for all} \quad i = 1, 2, 3. \tag{2.1.4}$$

(b) Commutating relations:

$$\begin{aligned} [\sigma_1, \sigma_2] &= 2i\sigma_3, \\ [\sigma_2, \sigma_3] &= 2i\sigma_1, \\ [\sigma_3, \sigma_1] &= 2i\sigma_2. \end{aligned} \tag{2.1.5}$$

(c) Anticommutating relations:

$$[\sigma_i, \sigma_j]_+ := \sigma_i\sigma_j + \sigma_j\sigma_i = 0, \quad \text{for all} \quad i, j = 1, 2, 3. \tag{2.1.6}$$

(d) Hermitian property

$$\sigma_i^* = \sigma_i, \quad \text{for} \quad i = 1, 2, 3. \tag{2.1.7}$$

In terms of the basis $\{\sigma_+, \sigma_-, \sigma_3\}$ these properties are written as

- Commutating relations:

$$\begin{aligned} [\sigma_+, \sigma_-] &= \sigma_3, \\ [\sigma_-, \sigma_3] &= 2\sigma_-, \\ [\sigma_3, \sigma_+] &= 2\sigma_+. \end{aligned} \tag{2.1.8}$$

- Anticommutating relations:

$$\begin{aligned} [\sigma_+, \sigma_-]_+ &= \mathbf{I}, \\ [\sigma_-, \sigma_3]_+ &= [\sigma_3, \sigma_+]_+ = 0, \end{aligned} \tag{2.1.9}$$

- σ_+ and σ_- are hermitian conjugate,

$$\sigma_+^* = \sigma_-. \tag{2.1.10}$$

Moreover, we have the following algebraic identities:

$$\begin{aligned} \sigma_1 \cdot \sigma_3 \cdot \sigma_1 &= -\sigma_3, \\ \sigma_1 \cdot \sigma_+ \cdot \sigma_1 &= \sigma_-, \\ \sigma_1 \cdot \sigma_- \cdot \sigma_1 &= \sigma_+, \end{aligned} \tag{2.1.11}$$

and

$$\begin{aligned} \sigma_2 \cdot \sigma_3 \cdot \sigma_2 &= -\sigma_3, \\ \sigma_2 \cdot \sigma_+ \cdot \sigma_2 &= -\sigma_-, \\ \sigma_2 \cdot \sigma_- \cdot \sigma_2 &= -\sigma_+. \end{aligned} \tag{2.1.12}$$

As consequence of these identities, also we deduce for every $X \in \mathfrak{sl}(2\mathbb{C})$,

$$\sigma_i X \sigma_i \in \mathfrak{sl}(2\mathbb{C}), \quad \text{for} \quad i = 1, 2, 3. \tag{2.1.13}$$

Consider now the Lie group

$$\mathbf{GL}(2, \mathbb{C}) = \{X \in M(n, \mathbb{C}) |\, \det X \neq 0\}. \tag{2.1.14}$$

Denoted by $\mathbf{SL}(2, \mathbb{C})$, the *special linear group* consisting of all matrices with determinant equal 1,

$$\mathbf{SL}(2, \mathbb{C}) = \{\mathbf{A} \in \mathbf{GL}(2, \mathbb{C}) |\, \det \mathbf{A} = 1\}, \tag{2.1.15}$$

is a $\mathbf{SL}(2, \mathbb{C})$ is a connected Lie subgroup of $\mathbf{GL}(2, \mathbb{C})$ [6]. Next we prove that the Lie algebra of $\mathbf{SL}(2, \mathbb{C})$ is precisely $\mathfrak{sl}(2, \mathbb{C})$.

Lemma 2.1.1. *For any $X \in \mathbf{gl}(2, \mathbb{C})$, we have*

$$\det \mathrm{e}^X = \mathrm{e}^{\mathrm{tr}(X)}. \tag{2.1.16}$$

Proof. □

Proposition 2.1.2. *We let consider the Lie subgroup* $\mathbf{SL}(2,\mathbb{C})$*. If* $\mathfrak{g} \subset \mathbf{gl}(2,\mathbb{C})$ *is its Lie algebra, then*

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}). \tag{2.1.17}$$

Proof. $\mathfrak{g}$ is the matrix Lie algebra of a Lie group $G \subset \mathbf{GL}(n,\mathbb{F})$, if for each $X \in \mathfrak{g}$ we have

$$\exp(X) \in G. \tag{2.1.18}$$

Let $X \in \mathfrak{sl}(2,\mathbb{C})$, then by Lemma (2.1.1) we have

$$\det \exp(X) = \exp(\operatorname{tr} X) = 1, \tag{2.1.19}$$

hence, $\exp(X) \in \mathbf{SL}(2,\mathbb{C})$. □

Since $\mathfrak{sl}(2,\mathbb{C})$ is a complex Lie algebra, then, by Proposition (1.2.2) we get

$$\exp(\mathfrak{sl}(2,\mathbb{C})) = G^0. \tag{2.1.20}$$

Here G^0 is the maximal connected Lie subgroup whose Lie algebra is $\mathfrak{sl}(2,\mathbb{C})$, thus it follows that $G^0 \subset \mathbf{SL}(2,\mathbb{C})$.

2.1.1 Involution Relations

Here follow [7] we introduce the involution relation for elements of $\mathfrak{sl}(2,\mathbb{C})$. Given $\xi, \eta, \lambda \in \mathbb{C}$, consider the matrix function $\mathbb{C} \ni \lambda \mapsto \mathcal{H}(\lambda) \in \mathfrak{sl}(2\mathbb{C})$ defined by

$$\mathcal{H}(\lambda) \stackrel{\text{def}}{=} \xi\sigma_+ + \eta\sigma_- + \frac{\lambda}{2i}\sigma_3. \tag{2.1.21}$$

Definition 2.1.1. *We say that* $\mathcal{H}(\lambda)$ *satisfies the involution property with respect to* σ *if*

$$\sigma_i\mathcal{H}(\overline{\lambda})\sigma_i = \overline{\mathcal{H}}(\lambda) \quad \forall \lambda \in \mathbb{C}, \quad i = 1, 2., \tag{2.1.22}$$

Observe that by property (2.1.13), we have that $\sigma_i\mathcal{H}(\overline{\lambda})\sigma_i \in \mathfrak{sl}(2,\mathbb{C})$ for $i = 1, 2$.

Proposition 2.1.3. *Condition (2.1.22) holds if and only if* $\xi = \epsilon\overline{\eta}$,

$$\mathcal{H}(\lambda) = \begin{pmatrix} \frac{\lambda}{2i} & \epsilon\overline{\eta} \\ \eta & -\frac{\lambda}{2i} \end{pmatrix}, \tag{2.1.23}$$

where λ *and* η *are arbitrary complex numbers and*

$$\epsilon = \begin{cases} 1 & if \quad \sigma = \sigma_1 \\ -1 & if \quad \sigma = \sigma_2 \end{cases} \tag{2.1.24}$$

Proof. By (2.1.21) we get

$$\overline{\mathcal{H}}(\lambda) = \overline{\xi}\sigma_+ + \overline{\eta}\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3. \tag{2.1.25}$$

It follows from here that letting $\sigma = \sigma_1$, condition (2.1.22) is written as follows,

$$\sigma_1\mathcal{H}(\overline{\lambda})\sigma_1 = \eta\sigma_+ + \xi\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3. \tag{2.1.26}$$

Thus we get $\xi = \overline{\eta}$. Now, for $\sigma = \sigma_2$ (2.1.22) is reduced to

$$\sigma_2\mathcal{H}(\overline{\lambda})\sigma_2 = -\eta\sigma_+ - \xi\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3, \tag{2.1.27}$$

then we obtain $\xi = -\overline{\eta}$. □

The involution property can be extended to the elements Lie group $\mathbf{SL}(2,\mathbb{C})$. Let $\mathcal{H}(\lambda)$ the function defined by (2.1.21). Define

$$\mathbf{A}(\lambda) \overset{\text{def}}{=} \exp \mathcal{H}(\lambda) \in \mathbf{SL}(2,\mathbb{C}) \tag{2.1.28}$$

Proposition 2.1.4. *If $\mathcal{H}(\lambda)$ satisfies (2.1.22), then we get*

$$\sigma \mathbf{A}(\overline{\lambda})\sigma = \overline{\mathbf{A}}(\lambda), \forall \lambda \in \mathbb{C}. \tag{2.1.29}$$

Proof.

$$\begin{aligned} \sigma \mathbf{A}(\overline{\lambda})\sigma &= \sigma \exp\left(\mathcal{H}(\overline{\lambda})\right)\sigma && (2.1.30)\\ &= \exp\left(\sigma\mathcal{H}(\overline{\lambda})\sigma\right) && (2.1.31)\\ &= \exp\left(\overline{\mathcal{H}}(\lambda)\right) \\ &= \overline{\exp\left(\mathcal{H}(\lambda)\right)} \\ &= \overline{\mathbf{A}}(\lambda). && (2.1.32) \end{aligned}$$

□

We can formulate a result analogous to Proposition (2.1.3) for matrices in the group $\mathbf{SL}(2,\mathbb{C})$, whit the involution relation.

Proposition 2.1.5. *Let $\lambda \mapsto \mathbf{A}(\lambda) \in \mathrm{SL}(2,\mathbb{C})$ be a function of λ satisfying the involution property (2.1.32). Then*

$$\mathbf{A}(\lambda) = \begin{pmatrix} a(\lambda) & \varepsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix}, \tag{2.1.33}$$

where ε is defined by (2.1.24).

Proof. Let

$$\mathbf{A}(\lambda) = \begin{pmatrix} a(\lambda) & c(\lambda) \\ b(\lambda) & d(\lambda) \end{pmatrix},$$

and

$$\overline{\mathbf{A}}(\lambda) = \begin{pmatrix} \overline{a}(\lambda) & \overline{c}(\lambda) \\ \overline{b}(\lambda) & \overline{d}(\lambda) \end{pmatrix}. \tag{2.1.34}$$

$$\sigma_1 \mathbf{A}(\overline{\lambda})\sigma_1 = \begin{pmatrix} d(\overline{\lambda}) & b(\overline{\lambda}) \\ c(\overline{\lambda}) & a(\overline{\lambda}) \end{pmatrix}, \tag{2.1.35}$$

and

$$\sigma_2 \mathbf{A}(\overline{\lambda})\sigma_2 = \begin{pmatrix} d(\overline{\lambda}) & -b(\overline{\lambda}) \\ -c(\overline{\lambda}) & a(\overline{\lambda}) \end{pmatrix}. \tag{2.1.36}$$

Comparing (2.1.34) with (2.1.35) and (2.1.34) with (2.1.36) we obtain

$$d(\lambda) = \overline{a}(\overline{\lambda}), \quad c(\lambda) = \varepsilon\overline{b}(\overline{\lambda}).$$

□

2.2 Linear Problem in $\mathfrak{sl}(2,\mathbb{C})$ with Involution Property

Let us consider the following linear problem,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x,\lambda)\mathbf{f}; \tag{2.2.1}$$

where $\mathbf{U}(x,\lambda)$ is a matrix function of the form

$$\mathbf{U}(x,\lambda) = \sqrt{\kappa}\,\overline{\psi}(x)\sigma_+ + \sqrt{\kappa}\psi(x)\sigma_- + \frac{\lambda}{2i}\sigma_3. \tag{2.2.2}$$

where ψ is a function in $x \in \mathbb{R}$, λ is a complex parameter and $\kappa \in \mathbb{R}$. We have $\operatorname{tr}\mathbf{U}(x,\lambda) = 0$ and hence $\mathbf{U}(x,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$. It is clear that $\mathbf{U}(x,\lambda)$ has the representation (2.1.23), where $\eta = \sqrt{\kappa}\psi$ and and $\epsilon = \operatorname{sign}(\kappa)$. Then by Proposition (2.1.3) the matrix function $\mathbf{U}$ satisfies the involution property

$$\sigma\mathbf{U}(x,\overline{\lambda})\sigma = \overline{\mathbf{U}}(x,\lambda) \tag{2.2.3}$$

where if $\kappa > 0$ then $\sigma = \sigma_1$ and if $\kappa < 0$ then $\sigma = \sigma_2$.

We shall assume that ψ is a C^∞ function in x and bounded on $\mathbb{R}$. Then the fundamental solution $\mathbf{T}(x,y)$ of (2.2.1)

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T} = \mathbf{U}(x,\lambda)\mathbf{T}, \tag{2.2.4}$$

$$\mathbf{T}|_{x=y} = \mathbf{I}. \tag{2.2.5}$$

is well defined for all $(x,y) \in \mathbb{R}^2$. Proposition (1.1.1) implies that $\mathbf{T}(x,y,\lambda)$ is in $\mathbf{SL}(2,\mathbb{C})$.

Proposition 2.2.1. *The fundamental solution $\mathbf{T}(x,y,\lambda)$ of the linear problem (2.2.4) satisfies the involution property*

$$\sigma\mathbf{T}(x,y,\overline{\lambda})\sigma = \overline{\mathbf{T}}(x,y,\lambda), \tag{2.2.6}$$

Proof. Condition (2.2.6) can be written as follow

$$\mathbf{T}(x,y,\lambda) = \sigma\overline{\mathbf{T}}(x,y,\overline{\lambda})\sigma, \tag{2.2.7}$$

We only need to prove that the expression in the right hand side of (2.2.7) is also the fundamental solution. Using (2.1.4), we derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma\overline{\mathbf{T}}(x,y,\overline{\lambda})\sigma\right) &= \sigma\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathbf{T}}(x,y,\overline{\lambda})\right)\sigma,\\
&= \sigma\overline{\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T}(x,y,\overline{\lambda})\right)}\sigma,\\
&= \sigma\overline{\mathbf{U}}(x,\overline{\lambda})\overline{\mathbf{T}}(x,y,\overline{\lambda})\sigma,\\
&= \sigma\overline{\mathbf{U}}(x,\overline{\lambda})\sigma\cdot\sigma\overline{\mathbf{T}}(x,y,\overline{\lambda})\sigma,\\
&= \mathbf{U}(x,\overline{\lambda})\sigma\overline{\mathbf{T}}(x,y,\overline{\lambda})\sigma,
\end{aligned}$$

Taking into account

$$\sigma\overline{\mathbf{T}}(x,x,\overline{\lambda})\sigma = \sigma\overline{\mathbf{I}}\sigma = \mathbf{I},$$

by definition of the fundamental solution, we get (2.2.7). □

2.2.1 Quasi-periodic case.

Assume that $\mathbf{U}$ is quasi-periodic i.e, there exist a real number $L > 0$, and a matrix $\mathbf{Q} \in \mathrm{SL}(2, \mathbb{C})$, such that

$$\mathbf{U}(x + 2L, \lambda) = \mathbf{Q}^{-1}\mathbf{U}(x, \lambda)\mathbf{Q}.$$

Since monodromy matrix is defined by

$$\mathbf{M}(\lambda) = \mathbf{T}(y + 2L, y, \lambda), \tag{2.2.8}$$

it follows immediately from Proposition (2.2.1) that the monodromy matrix satisfies involution property.

Proposition 2.2.2. *The monodromy matrix takes the form*

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \varepsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix}, \tag{2.2.9}$$

where

$$\epsilon = \begin{cases} 1 & \textit{if } \kappa > 0 \\ -1 & \textit{if } \kappa < 0 \end{cases} \tag{2.2.10}$$

Proof. This result follows immediately from proposition (2.1.5) because the monodromy matrix is in the group $\mathbf{SL}(2, \mathbb{C})$ for each $\lambda \in \mathbb{C}$, and satisfy the involution property. □

The complex functions $a(\lambda)$ and $b(\lambda)$ are called the ***transition coefficients*** [7]. Since $\mathbf{M}(\lambda) \in \mathbf{SL}(2, \mathbb{C})$, for each real λ the transition coefficient satisfy the normalization condition

$$|a(\lambda)|^2 - \varepsilon|b(\lambda)|^2 = 1. \tag{2.2.11}$$

Riccati's Equations and Asymptotic Series

Now, we consider the linear problem (2.2.4)-(2.2.5). Observe that $\mathbf{U}(x, \lambda)$ (2.2.1) can be represented as follows:

$$\mathbf{U}(x, \lambda) = \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3,$$

with

$$\mathbf{U}_0(x) = \sqrt{\kappa}\begin{pmatrix} 0 & \overline{\psi}(x) \\ \psi(x) & 0 \end{pmatrix}.$$

We assume that $\mathbf{U}$ has the quasiperiodicity condition (2.2.8) with

$$\mathbf{Q} = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{2.2.12}$$

where $\theta \in \mathbb{R}$ is a constant.

Proposition 2.2.3. *The quasiperiodic condition (*2.2.8*) for the matrix coefficient* $\mathbf{U}(x, \lambda)$*, with* $\mathbf{Q}$ *in (*2.2.12*) is equivalent to the following condition for* ψ*:*

$$\psi(x + 2L) = \psi(x)e^{i\theta}. \tag{2.2.13}$$

Proof. Since σ_3 and $\mathbf{Q}$ commute.

$$\begin{aligned}
\mathbf{Q}^{-1}\mathbf{U}(x+2L)\mathbf{Q} &= \mathbf{Q}^{-1}(\mathbf{U}_0(x+2L)+\frac{\lambda}{2i}\sigma_3)\mathbf{Q} \\
&= \mathbf{Q}^{-1}(\mathbf{U}_0(x+2L)\mathbf{Q}+\frac{\lambda}{2i}\sigma_3) \\
&= (\mathbf{U}_0(x)+\frac{\lambda}{2i}\sigma_3).
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}^{-1}(\mathbf{U}_0(x+2L))\mathbf{Q} &= \sqrt{\kappa}\begin{pmatrix} \mathrm{e}^{-\frac{i\theta}{2}} & 0 \\ 0 & \mathrm{e}^{\frac{i\theta}{2}} \end{pmatrix}\begin{pmatrix} 0 & \overline{\psi}(x+2L) \\ \psi(x+2L) & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\frac{i\theta}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{i\theta}{2}} \end{pmatrix} \\
&= \sqrt{\kappa}\begin{pmatrix} 0 & \overline{\psi}(x+2L)\mathrm{e}^{-i\theta} \\ \psi(x+2L)\mathrm{e}^{i\theta} & 0 \end{pmatrix}
\end{aligned}$$

□

Theorem 2.2.4. *The fundamental solution* $\mathbf{T}(x,y,\lambda)$ *of the system* (2.2.4)*-*(2.2.5) *has the following representation:*

$$\mathbf{T}(x,y,\lambda) = (\mathbf{I}+\mathbf{W}(x,\lambda))\cdot\exp(\mathbf{Z}(x,y,\lambda))\cdot(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1}. \tag{2.2.14}$$

Here

- $\mathbf{W}$ *is an anti-diagonal matrix which satisfy the Riccati equation*

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}x}+i\lambda\sigma_3\mathbf{W}+\mathbf{W}\mathbf{U_0}\mathbf{W}-\mathbf{U_0}=0 \tag{2.2.15}$$

.

- $\mathbf{Z}$ *is a diagonal matrix such that*

$$\mathbf{Z}(x,y,\lambda) = \frac{(x-y)\lambda}{2i}\sigma_3+\int_y^x \mathbf{U_0}(z)\mathbf{W}(z,\lambda)\mathrm{d}z, \tag{2.2.16}$$

$$\mathbf{Z}(x,x,\lambda) = \mathbf{0}. \tag{2.2.17}$$

- $\mathbf{W}$ *and* $\mathbf{Z}$ *has the following asymptotic representations as* $|\lambda|\to\infty$*:*

$$\mathbf{W}(x,\lambda) = \sum_{n=1}^{\infty}\frac{\mathbf{W}_n(x)}{\lambda^n}+O(|\lambda|^{-\infty}), \tag{2.2.18}$$

$$\mathbf{Z}(x,y,\lambda) = \frac{(x-y)\lambda}{2i}\sigma_3+\sum_{n=1}^{\infty}\frac{\mathbf{Z}_n(x,y)}{\lambda^n}+O(|\lambda|^{-\infty}). \tag{2.2.19}$$

Proof. Putting the expression (2.2.14) into (2.2.4), we get

$$\frac{d\mathbf{T}}{dx} = \left(\frac{d\mathbf{W}}{dx}\exp(\mathbf{Z})+(\mathbf{I}+\mathbf{W})\exp(\mathbf{Z})\frac{\partial\mathbf{Z}}{\partial x}\right)(\mathbf{I}+\mathbf{W}(y,\lambda)^{-1}). \tag{2.2.20}$$

On other hand, we have

$$\mathbf{U}\mathbf{T} = (\mathbf{U_0}+\lambda\mathbf{U_1})(\mathbf{I}+\mathbf{W}(x,\lambda))\cdot\exp(\mathbf{Z}(x,y,\lambda))\cdot(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1}, \tag{2.2.21}$$

comparing (2.2.20) with (2.2.21), the matrix $(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1}$, which is x-independent, is canceled and it is possible to split the result into diagonal and anti-diagonal parts. Thus we obtain

$$\frac{d\mathbf{W}}{dx}+\mathbf{W}\frac{\partial \mathbf{Z}}{\partial x}=\mathbf{U}_0+\lambda\mathbf{U}_1\mathbf{W}, \tag{2.2.22}$$

$$\frac{\partial \mathbf{Z}}{\partial x}=\mathbf{U}_0\mathbf{W}+\lambda\mathbf{U}_1. \tag{2.2.23}$$

Substituting (2.2.23) into (2.2.22) and using the fact that $\mathbf{U}_1$ anticommutes with $\mathbf{W}$, we obtain the Riccati type equation (2.2.15). Now, we must note that the boundary condition (2.2.5) implies $\mathbf{Z}(x,y,\lambda)=\mathbf{0}$, and therefore equation (2.2.23) can be easily solved and we get equation (2.2.16).

Now let us suppose that

$$\mathbf{W}(x,\lambda)=\sum_{n=1}^{\infty}\frac{\mathbf{W}_n(x)}{\lambda^n},$$

satisfies de Riccati equation (2.2.15). Then we have

$$\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\frac{d\mathbf{W_n}}{dx}+\sum_{k=1}^{\infty}i\sigma_3\frac{\mathbf{W}_k}{\lambda^{k-1}}+\left(\sum_{m=1}^{\infty}\frac{\mathbf{W}_m(x)}{\lambda^m}\right)\mathbf{U}_0\left(\sum_{s=1}^{\infty}\frac{\mathbf{W}_s(x)}{\lambda^x}\right)=0 \tag{2.2.24}$$

Note that we can simplify the third term in the left hand side of (2.2.24):

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\frac{\mathbf{W}_m(x)}{\lambda^m}\right)\mathbf{U}_0\left(\sum_{s=1}^{\infty}\frac{\mathbf{W}_s(x)}{\lambda^x}\right) &= \sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\mathbf{W}_m\mathbf{U}_0\mathbf{W}_s}{\lambda^{m+s}},\\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{s=r-1}\frac{\mathbf{W}_{r+1-s}\mathbf{U}_0\mathbf{W}_s}{\lambda^{r+1}},\end{aligned}$$

where $r=m+s-1$. thus, we take $q=k-1$ and then we can rewrite equation (2.2.24) as follow

$$i\sigma_3\mathbf{W}_1-\mathbf{U}_0+\frac{1}{\lambda}\left(\frac{d\mathbf{W}_1}{dx}+i\sigma_3\mathbf{W}_2\right)+\sum_{n=2}^{\infty}\frac{1}{\lambda^n}\left(\frac{d\mathbf{W}_n}{dx}+i\sigma_3\mathbf{W}_{n+1}+\sum_{k=1}^{n-1}\mathbf{W}_k\mathbf{U}_0\mathbf{W}_{n-k}\right)=0.$$

Since this equation holds for all $\lambda\in\mathbb{R}-\{0\}$ and all natural numbers $n\geq 2$, then we have the following set of recursion relations:

$$\begin{aligned}\mathbf{W}_1(x) &= -i\sigma_3\mathbf{U}_0(x),\\ \mathbf{W}_2(x) &= i\sigma_3\frac{d\mathbf{W}_1(x)}{dx},\\ \mathbf{W}_{n+1}(x) &= i\sigma_3\left(\frac{d\mathbf{W}_n(x)}{dx}+\sum_{k=1}^{n-1}\mathbf{W}_k(x)\mathbf{U}_0(x)\mathbf{W}_{n-k}(x)\right).\end{aligned}$$

Therefore, the coefficients of the asymptotic series for $\mathbf{W}(x,\lambda)$ can be expressed, locally, in terms of $\mathbf{U}_0$ and its derivatives. Now, taking into account the map $\mathbf{Z}(x,y,\lambda)$ satisfies the equation (2.2.16), which depends on the map $\mathbf{W}(x,\lambda)$, it implies the existence of its asymptotic expansion. □

Proposition 2.2.5. *The matrix diagonal map $\mathbf{W}(x,\lambda)$, which appears as factor in the fundamental solution $\mathbf{T}(x,y,\lambda)$, has the following properties:*

i) Involution property

$$\overline{\mathbf{W}}(x,\lambda) = \sigma \mathbf{W}(x,\overline{\lambda})\sigma.$$

ii) Quasi-periodicity,

$$\mathbf{W}(x+2L,\lambda) = \mathbf{Q}^{-1}\mathbf{W}(x,\lambda)\mathbf{Q}.$$

Proof. i) Using the relations (2.1.11)-(2.1.12), it is enough to prove the involution property for the coefficients of the asymptotic expansion for $\mathbf{W}(x,\lambda)$.

$$\sigma\mathbf{W}_1(x)\sigma = \sigma(-i\sigma_3\mathbf{U}_0(x))\sigma = i\sigma_3(\sigma\mathbf{U}_0(x)\sigma) = i\sigma_3\overline{\mathbf{U}}_0(x) = \overline{\mathbf{W}}_1(x),$$

$$\sigma\mathbf{W}_2(x)\sigma = \sigma\left(i\sigma_3\frac{d\mathbf{W}_1(x)}{dx}\right)\sigma = -i\sigma_3\left(\frac{d\overline{\mathbf{W}}_1(x)}{dx}\right) = \overline{\mathbf{W}}_2(x).$$

Now, assuming that $\mathbf{W}_k$ has the involution property for all positive integer $k \leq n$, by induction, one can deduce immediately that the matrix

$$\mathbf{W}_{n+1}(x) = \sigma_3\left(\frac{d\mathbf{W}_n(x)}{dx} + \sum_{k=1}^{n-1}\mathbf{W}_k(x)\mathbf{U}_0(x)\mathbf{W}_{n-k}(x)\right),$$

also possess this property.

ii) In order to prove that $\mathbf{W}$ is quasi-periodic, first we note both σ_3 and $\mathbf{Q} = \exp\{\frac{i\theta}{2}\sigma_3\}$ are diagonal matrices and hence they commute. Since the matrix map $\mathbf{U}(x,\lambda)$ is quasiperiodic, $\mathbf{U}_0(x)$ also has this property. Now, using the recursion relation for the coefficient of the asymptotic expansion of $\mathbf{W}(x,\lambda)$, we get

$$\begin{aligned}
\mathbf{W}_1(x+2L) &= -i\sigma_3\mathbf{U}_0(x+2L) = \mathbf{Q}^{-1}(-i\sigma_3\mathbf{U}_0(x))\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{W}_1(x)\mathbf{Q},\\
\mathbf{W}_2(x+2L) &= i\sigma_3\frac{d\mathbf{W}_1(x+2L)}{dx} = \mathbf{Q}^{-1}(i\sigma_3\frac{d\mathbf{W}_1(x)}{dx})\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{W}_2(x)\mathbf{Q}.
\end{aligned}$$

Then again by induction. Suppose that the coefficient $\mathbf{W}_k(x)$ of the asymptotic serie are quasi-periodic for all positive integer $k \leq n$. we conclude that the coefficient $\mathbf{W}_{n+1}(x)$ is also quasi-periodic. as a consequence, $\mathbf{W}(x,\lambda)$ has the quasi-periodicity property.

□

Taking into account the properties of the anti-diagonal matrix map $\mathbf{W}(x,\lambda)$ (which has been used in the Proposition (2.2.4) and (2.2.5)) we derive the following result.

Definition 2.2.1. *We say that a complex valued function $u(x)$ which depends on the real parameter x satisfies the quasi-periodicity condition, if there exists a real number $L > 0$ and $0 \leq \theta < 2\pi$ such that $u(x+2L) = \mathrm{e}^{i\theta}u(x)$*

Corollary 2.2.6. *The matrix map $\mathbf{W}(x,\lambda)$ can be expressed by the formula*

$$\mathbf{W}(x,\lambda) = i\sqrt{\kappa}(w(x,\lambda)\sigma_- - \overline{w}(x,\lambda)\sigma_+), \tag{2.2.25}$$

where $w(x,\lambda)$ is a complex valued function, differentiable on x and analytic in λ, with the following properties:

i) Asymptotic expansion

$$w(x,\lambda) = \sum_{n=1}^{\infty} \frac{w_n(x)}{\lambda^n}, \tag{2.2.26}$$

with

$$\begin{aligned} w_1(x) &= \psi(x,t), \\ w_2(x) &= -i\frac{\partial \psi(x,t)}{\partial x}, \\ w_{n+1}(x) &= -i\frac{dw_n(x)}{dx} + \kappa\overline{\psi}(x,t)\sum_{k=1}^{n-1} w_k w_{n-k}. \end{aligned}$$

ii) $w(x,\lambda)$ satisfies the property

$$w(x+2L,\lambda) = \mathrm{e}^{i\theta} w(x,\lambda). \tag{2.2.27}$$

Proof. Since de matrix map $\mathbf{W}(x,\lambda)$ belongs to $\mathbf{SL}(2,\mathbb{C})$,it is anti-diagonal and satisfies the involution property. The existence of the complex valued function $w(x,\lambda)$ that satisfies equation (2.2.25) is guaranteed by the algebraic Lemma (2.1.5). Properties i) and ii) are direct consequences of Propositions (2.2.4) and (2.2.5). □

Corollary 2.2.7. *The monodromy matrix $\mathbf{M}(\lambda)$ has the following representation*

$$\mathbf{M}(\lambda) = (\mathbf{I} + \mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda))(\mathbf{I} + \mathbf{W}(-L,\lambda))^{-1},$$

where

$$\mathbf{Z}_L(\lambda) = -i\lambda L\sigma_3 + \int_{-L}^{L} \mathbf{U}_0 \mathbf{W}(x,\lambda)dx, \tag{2.2.28}$$

*and $\mathbf{W}(x,\lambda)$ is defined as in the Theorem (*2.2.4*).*

Time evolution and Integrals of motion

In Section 1.4 of Chapter 1, we conclude that the determinant and the trace of the monodromy matrix are time independent. Hence, they are the integrals of motion of the zero curvature equation. Our purpose here is to write these time-invariants in terms of the coefficient which defined the matrix function $\mathbf{U}(x,t,\lambda)$ (2.2.1).

Theorem 2.2.8. *The trace of the monodromy matrix $\mathbf{M}(\lambda)$ takes the form*

$$F_L(\lambda) = \mathrm{tr}(\mathbf{M}(\lambda)\mathbf{Q}(\theta)) = 2\cos(\varphi_L(\lambda) + \frac{\theta^2}{2} - \lambda L), \tag{2.2.29}$$

where $\varphi_L(\lambda)$ is given by

$$\varphi_L(\lambda) = \kappa \int_{-L}^{L} \overline{\psi}(x,t) w(x,\lambda) dx, \tag{2.2.30}$$

and has the following properties:

1. $\varphi_L(\lambda) = \overline{\varphi}_L(\overline{\lambda})$.
2. $\varphi_L(\lambda) = \kappa \sum_{n=1}^{\infty} \frac{I_n}{\lambda^n} + O(|\lambda|^{-\infty})$, *with*

$$I_n = \int_{-L}^{L} \overline{\psi}(x,t) w_n(x). \tag{2.2.31}$$

Proof. Before to give a complete proof of the theorem, we have to state some preliminary facts.

Since the matrix map $\mathbf{W}(x,\lambda)$ is quasi-periodic, the matrix $(\mathbf{I}+\mathbf{W}(x,\lambda))$ and its inverse are also quasi-periodic:

$$\begin{aligned}\mathbf{I}+\mathbf{W}(x+2L,\lambda) &= \mathbf{I}+\mathbf{Q}^{-1}\mathbf{W}(x,\lambda)\mathbf{Q}\\ &= \mathbf{Q}^{-1}(\mathbf{I}+\mathbf{W}(x,\lambda))\mathbf{Q}.\end{aligned}$$

Using this fact and theorem (2.2.7), we find

$$\mathbf{M}(\lambda)\mathbf{Q}(\theta) = (\mathbf{I}+\mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda))\mathbf{Q}(\theta)(\mathbf{I}+\mathbf{W}(L,\lambda))^{-1}, \tag{2.2.32}$$

The commutativity of σ_3 and $\mathbf{Z}_L(\lambda)$ gives

$$F_L(\lambda) = \operatorname{tr}\mathbf{M}(\lambda)\mathbf{Q}(\theta) = \operatorname{tr}\exp\left\{\mathbf{Z}_L(\lambda)+\frac{i\theta}{2}\sigma_3\right\}. \tag{2.2.33}$$

Remark that $\mathbf{M}(\lambda)\mathbf{Q}(\theta)$ has determinant equal to 1. Then,

$$\operatorname{tr}\mathbf{Z}_L(\lambda) = O(|\lambda|^{-\infty}). \tag{2.2.34}$$

Now, we notice that the following fact. If u and v are two quasi-periodic complex functions then $\overline{u}v$ is a periodic function; indeed, $\overline{u}(x+2L)v(x+2L) = \overline{e^{i\theta}u(x)}e^{i\theta}v(x) = u(x)v(x)$. After this observation we can see that the diagonal matrix $\mathbf{U}_0\mathbf{W}$ involved in the integral equation (2.2.28),

$$\mathbf{U}_0(x)\mathbf{W}(x,\lambda) = i\kappa\begin{pmatrix}\overline{\psi}(x)w(x,\lambda) & 0\\ 0 & -\psi(x)\overline{w}(x,\lambda)\end{pmatrix}, \tag{2.2.35}$$

is periodic. Therefore, the integral in (2.2.28) is independent of the choice of the fundamental domain. We define the function

$$\varphi_L(\lambda) := \kappa\int_{-L}^{L}\overline{\psi}(x)w(x,\lambda)dx. \tag{2.2.36}$$

Taking into account that the function $w(x,\lambda)$ has the asymptotic series in (2.2.26), we obtain

$$\varphi_L(\lambda) = \kappa\sum_{n=1}^{\infty}\frac{I_n}{\lambda^n} + O(|\lambda|^{-\infty}), \tag{2.2.37}$$

with

$$I_n(\psi,\overline{\psi}) = \int_{-L}^{L}\overline{\psi}(x)w_n(x). \tag{2.2.38}$$

Moreover, it follows from here that series (2.2.37) has real coefficients, and hence

$$\varphi_L(\lambda) = \overline{\varphi}_L(\overline{\lambda}), \tag{2.2.39}$$

from the fact $\mathbf{U}_0\mathbf{W}$ belongs to $\mathbf{SL}(2,\mathbb{C})$. Thus, $\mathbf{Z}_L(\lambda)$ can be expressed as

$$\mathbf{Z}_L(\lambda) = i\sigma_3(\varphi_L(\lambda)-\lambda L), \tag{2.2.40}$$

and therefore

$$F_L(\lambda) = 2\cos\left(\varphi_L(\lambda)+\frac{\theta}{2}-\lambda L\right) \tag{2.2.41}$$

□

For each $n = 1, 2, ...$, the functionals I_n, (2.2.38) are the local integral of motion for the quasi-periodic case. Using the formulas for the coefficients of the asymptotic expansion of $w(x)$ (2.2.26) we can express the first four integral of motion in terms of $\psi(x)$:

$$\begin{aligned}
I_1 &= \int_{-L}^{L} |\psi(x)|^2 dx, \\
I_2 &= \int_{-L}^{L} \left(-i\bar{\psi}(x)\frac{d}{dx}\psi(x)\right) dx, \\
I_3 &= \int_{-L}^{L} \left(-\bar{\psi}(x)\frac{d^2}{dx^2}\psi(x) + \kappa|\psi(x)|^4\right) dx, \\
I_4 &= \int_{-L}^{L} i\left(\bar{\psi}(x)\frac{d^3}{dx^3}\psi(x) - \kappa|\psi(x)|^2\left(\psi(x)\frac{d}{dx}\bar{\psi}(x) + 4\bar{\psi}(x)\frac{d}{dx}\psi(x)\right)\right) dx.
\end{aligned}$$

2.2.2 Decreasing case.

Recall that the matrix coefficient of the linear problem (2.2.4) has the form $\mathbf{U}(x,\lambda) = \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3$. Let us assume that $\mathbf{U}_0(x) \in L_1^{2\times 2}(\mathbb{R}) \bigcap C^\infty(\mathbb{R})$, and λ is a real parameter. The assumption of integrability of $\mathbf{U}_0(x)$ implies that $\psi \in L_1(\mathbb{R})$. Moreover, this is a sufficient condition for the existence of the fundamental solution $\mathbf{T}(x, y, \lambda)$ of the linear problem (2.2.4)-(2.2.5). It follows from Proposition (1.3.5) that the limits

$$\mathbf{T}_\pm(x,\lambda) = \lim_{y\to\pm\infty} \mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda), \tag{2.2.42}$$

exist for each x and real λ. Here

$$\mathbf{E}(x,\lambda) = \exp\left(\frac{\lambda}{2i}(x)\sigma_3\right) \tag{2.2.43}$$

is the fundamental solution of the linear problem

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}(x,\lambda) &= \frac{\lambda}{2i}\sigma_3\mathbf{E}(x,\lambda) \\
\mathbf{E}(0,\lambda) &= \mathbf{I},
\end{aligned}$$

for each real λ.

We have by the Proposition (1.2.10) the integral representations for $\mathbf{T}_\pm(x,\lambda)$ are given

$$\mathbf{T}_-(x,\lambda) = \mathbf{E}(x,\lambda) + \int_{-\infty}^{x} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{T}_-(z,\lambda)\mathrm{d}z, \tag{2.2.44}$$

$$\mathbf{T}_+(x,\lambda) = \mathbf{E}(x,\lambda) - \int_{x}^{\infty} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{T}_+(z,\lambda)\mathrm{d}z,. \tag{2.2.45}$$

Moreover, by Proposition (1.3.5) it follows that the matrix function $\mathbf{T}_\pm(x,\lambda)$ satisfy the linear problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T}_\pm = \mathbf{U}(x,\lambda)\mathbf{T}_\pm, \tag{2.2.46}$$

with the asymptotic conditions

$$\mathbf{T}_\pm(x,\lambda) \to \mathbf{E}(x,\lambda) \quad \text{as} \quad x \to \pm\infty. \tag{2.2.47}$$

Follow [7] the functions $\mathbf{T}_{\pm}(x,\lambda)$ will be called *Jost solution* and play a significant role in posterior developments. We shall use the column notation for the matrix $\mathbf{T}_{\pm}(x,\lambda)$:

$$\mathbf{T}_{\pm}(x,\lambda) = (\mathbf{T}_{\pm}^{(1)}(x,\lambda), \mathbf{T}_{\pm}^{(2)}(x,\lambda)). \tag{2.2.48}$$

Let us examine the analytic properties for the elements of $\mathbf{T}_{\pm}(x,\lambda)$ considered as functions of λ for a fixed x. Recall that if the matrix $\mathbf{U}(x,\lambda)$ is analytic with respect the parameter λ, then $\mathbf{T}(x,y,\lambda)$ is an entire function. However, $\mathbf{T}_{\pm}(x,\lambda)$ are not analytic in general, because their definition involves a limit. Although, the columns of the $\mathbf{T}_{\pm}(x,\lambda)$ can be analytic extended into distinct half complex planes.

Proposition 2.2.9. *Let $\mathbf{T}_{\pm}(x,\lambda)$ be the functions defined by (2.2.42). Then,*

a) The column vectors $\mathbf{T}_{-}^{(1)}(x,\lambda)$ and $\mathbf{T}_{+}^{(2)}(x,\lambda)$ can be extended analytically to the upper half plane.

b) The column vectors $\mathbf{T}_{+}^{(1)}(x,\lambda)$ and $\mathbf{T}_{-}^{(2)}(x,\lambda)$ can be extended analytically to the lower half plane.

Proof. This result follows from of the integral representation (2.2.44)-(2.2.45) and the absolute integrability of $\mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)$. □

The monodromy matrix in the decreasing case is defined by

$$\mathbf{M}(\lambda) = \mathbf{T}_{+}^{-1}(x,\lambda)\mathbf{T}_{-}(x,\lambda). \tag{2.2.49}$$

Proposition 2.2.10. *The monodromy matrix $\mathbf{M}(\lambda)$ satisfies the involution property*

$$\overline{\mathbf{M}}(\lambda) = \sigma\mathbf{M}(\lambda)\sigma.$$

Moreover, there exist complex valued functions $a(\lambda)$,$b(\lambda)$ such that

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\bar{b}(\lambda) \\ b(\lambda) & \bar{a}(\lambda) \end{pmatrix}, \tag{2.2.50}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } \kappa > 0 \\ -1 & \text{if } \kappa < 0 \end{cases} \tag{2.2.51}$$

and

$$|a(\lambda)|^2 - \epsilon|b(\lambda)|^2 = 1. \tag{2.2.52}$$

Proof. We only have to check that $\mathbf{T}_{\pm}(x,\lambda)$ also satisfy the involution property. Since $\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda)$ converges uniformly to $\mathbf{T}_{\pm}(x,\lambda)$, as $y\to\pm\infty$, it only remains to verify that this sequence also satisfy the involution property.

$$\begin{aligned} \sigma\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda)\sigma &= \sigma\mathbf{T}(x,y,\lambda)\sigma\sigma\mathbf{E}(y,\lambda)\sigma \\ &= \overline{\mathbf{T}}(x,y,\lambda)\exp\{\frac{\lambda}{2i}\sigma\sigma_3\sigma\} \\ &= \overline{\mathbf{T}}(x,y,\lambda)\exp\{\frac{-\lambda}{2i}\sigma_3\} \\ &= \overline{(\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda))}. \end{aligned}$$

Therefore, the involution property also holds for $\mathbf{T}_\pm(x,\lambda)$. On the other hand,

$$\begin{aligned}
\overline{\mathbf{M}}(\lambda) &= \overline{(\mathbf{T}_+^{-1}(x,\lambda)\mathbf{T}_-(x,\lambda))} \\
&= (\overline{\mathbf{T}}_+(x,\lambda))^{-1}\overline{\mathbf{T}}_-(x,\lambda), \\
&= (\sigma\mathbf{T}_+^{-1}(x,\lambda)\sigma)\sigma\mathbf{T}_-(x,\lambda)\sigma, \\
&= \sigma\mathbf{M}(\lambda)\sigma.
\end{aligned}$$

The existence of the complex valued functions $a(\lambda), b(\lambda)$ is guaranteed by Lemma (2.1.5), because $\mathbf{M}(\lambda)$ has the involution property. These functions satisfy the normalized relation (2.2.52) because of unimodularity of the monodromy. □

Trace formulas for monodromy

Let us now discuss the local integrals of motion. We assume that $\psi(x)$, and $\overline{\psi}(x)$ are of Schwartz type. In order to exploit the results obtained earlier, suppose that $\psi(x)$, and $\overline{\psi}(x)$ are the limits, as $L \to \infty$ of the $2L$-periodic functions $\psi_L(x), \overline{\psi}_L(x)$.

In this case the densities $P_n(x) = \overline{\psi}(x)w(x,\lambda)$ of the local integral of the motion defined by the quasi-periodic case have limits as $L \to \infty$ which are also of Schwartz type.

Therefore, we can take the limit, as $L \to \infty$, and obtain

$$I_n = \int_{-\infty}^{\infty} \overline{\psi}(x)w(x,\lambda)\mathrm{d}x.$$

Here, each $P_n(x)$ is constructed from $\psi(x)$, and $\overline{\psi}(x)$ according to the formulae for the quasi-periodic case.

Let us now consider the limit, as $L \to \infty$, of the generating function $P_L(\lambda)$,

$$P_L(\lambda) = \arccos\left(\frac{1}{2}\operatorname{tr}\mathbf{M}_L(\lambda)\right).$$

The definition of the monodromy matrix in the quasiperiodic case implies that for real λ

$$\begin{aligned}
\operatorname{tr}\mathbf{M}_L &= \mathrm{e}^{-i\lambda L}a(\lambda) + \mathrm{e}^{i\lambda L}\overline{a}(\lambda) + o(1), \\
&= 2|a(\lambda)|\cos\left(\arg a(\lambda) - \lambda L\right) + o(1) \\
&\text{as} \quad L \to \infty.
\end{aligned}$$

Since $b(\lambda)$ is of Schwartz type, the normalization relation yields

$$|a(\lambda)| = 1 + O(|\lambda|^{-\infty}).$$

Thus, the generating function for the conservation laws in the limit $L \to \infty$, coincides with $\log a(\lambda)$.

$$\lim_{L\to\infty}\left(P_L(\lambda) + \lambda L\right) = \frac{1}{i}\log a(\lambda),$$

up to terms of order $O(|\lambda|^{-\infty})$. We deduce that $\log a(\lambda)$ is the generating function of local integral of motion

$$\log a(\lambda) = i\kappa\sum_{n=1}^{\infty}\frac{I_n}{\lambda^n} + O(|\lambda|^{-\infty}).$$

The coefficients I_n can be determined from the representation of $a(\lambda)$ trough $b(\lambda)$ and its zeros. Since $\log(1+\epsilon|b(\mu)|^2)$ in the expression for $a(\lambda)$ are of Schwartz type, then expanding $\frac{1}{\mu-\lambda}$ in a geometric progression leads to the asymptotic expansion

$$\log a(\lambda) = i\kappa\sum_{n=1}^{\infty}\frac{c_n}{\lambda^n} + O(|\lambda|^{-\infty}),$$

where

$$\begin{aligned} c_n &= \frac{1}{2\pi\kappa}\int_{-\infty}^{\infty}\log\left(1+\epsilon|b(\lambda)|^2\right)\lambda^{n-1}\mathrm{d}\lambda + \frac{1}{2n\kappa}\sum_{j=1}^{m}\left(\lambda_j^{-n}-\lambda_j^{n}\right), \\ n &= 1,2,..., \end{aligned}$$

Here, if $\epsilon = 1$, the sum over the zeros on the right hand side of the last equation disappears; therefore

$$c_n = I_n = \int_{-\infty}^{\infty} P_n(x)dx,$$

relating $\psi(x)$, and $\overline{\psi}(x)$ with the functionals of $b(\lambda), \overline{b}(\lambda)$ and $\lambda_j, \overline{\lambda}_j$. In spectral theory such formulae are called **trace identities** [7].

2.2.3 The Spectral Problem.

Consider the linear problem

$$\frac{d\mathbf{f}}{dx} = \mathbf{U}(x,\lambda)\mathbf{f}, \quad \mathbf{f}(x) = (f_1(x), f_2(x)) \in \mathbb{C}^2. \tag{2.2.53}$$

Let $\mathfrak{X} = (L_2(\mathbb{R})\otimes\mathbb{C}^2, \langle,\rangle)$ be a Hilbert space with inner product

$$\langle\mathbf{f}(x),\mathbf{g}(x)\rangle = \int_{-\infty}^{\infty}\mathbf{f}(x)\cdot\overline{\mathbf{g}}(x)\mathrm{d}x. \tag{2.2.54}$$

Multiplying both sides of (2.2.53) by $i\sigma_3$, it is easy to see that the linear problem 2.2.53 for $\lambda\in\mathbb{C}$ is equivalent to the eigenvalue problem in $\mathfrak{X}$

$$\mathscr{L}\mathbf{f} = \frac{\lambda}{2}\mathbf{f}, \tag{2.2.55}$$

for the first order matrix differential operator

$$\mathscr{L} = i\sigma_3\frac{d}{dx} + i\sqrt{\kappa}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+). \tag{2.2.56}$$

So λ is interpreted as a spectral parameter for $\mathscr{L}$. Let $\mathscr{L}^*$ the adjoint operator of $\mathscr{L}$,

$$\langle\mathscr{L}\mathbf{f}, g\rangle = \langle\mathbf{f},\mathscr{L}^* g\rangle. \tag{2.2.57}$$

Using properties of $\sigma_\pm$ (2.1.10), we get

$$\mathscr{L}^* = i\sigma_3\frac{d}{dx} + i\mathrm{sign}(\kappa)\sqrt{\kappa}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+). \tag{2.2.58}$$

It follows from here that

Case $\kappa \geq 0$. $\mathscr{L}$ is formally self-adjoint, $\mathscr{L} = \mathscr{L}^*$ and the eigenvalues λ are real. This implies that the analytic extension of function the $a(\lambda)$ has no zeros (see Proposition 2.2.15).

Case $\kappa < 0$. $\mathscr{L}$ is not self-adjoint and the function $a(\lambda)$ may have zeros, which correspond to the discrete spectrum of (2.2.55)

Analytic behavior of the transition coefficients

Now, we shall deduce some analytic properties of the transition coefficients and obtain integral representation for them, which will be useful in the study of time evolution. Recall

$$\mathbf{T}_\pm(x,\lambda) = (\mathbf{T}_\pm^{(1)}(x,\lambda), \mathbf{T}_\pm^{(2)}(x,\lambda)). \tag{2.2.59}$$

Observe that the columns of the matrix functions $\mathbf{T}_\pm$ are the eigenfunctions of eigenvalue problem (2.2.55).

In order to study the analytic behavior of the transition coefficients, we need to prove the following result.

Lemma 2.2.11. *The transition coefficients have the following expressions*

$$\begin{aligned} a(\lambda) &= \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)), & (2.2.60)\\ b(\lambda) &= \det(\mathbf{T}_+^{(1)}(x,\lambda), \mathbf{T}_-^{(1)}(x,\lambda)). & (2.2.61)\end{aligned}$$

Proof. Let us denote

$$\mathbf{T}_\pm^{(1)} = \begin{pmatrix} \mathbf{T}_\pm^{(1)1} \\ \mathbf{T}_\pm^{(1)2} \end{pmatrix} \quad \text{and} \quad \mathbf{T}_\pm^{(2)} = \begin{pmatrix} \mathbf{T}_\pm^{(2)1} \\ \mathbf{T}_\pm^{(2)2} \end{pmatrix}.$$

With this notation, we can express the monodromy matrix (2.2.49) as

$$\mathbf{M}(\lambda) = \begin{pmatrix} \mathbf{T}_+^{(2)2} & -\mathbf{T}_+^{(2)1} \\ -\mathbf{T}_+^{(1)2} & \mathbf{T}_+^{(1)1} \end{pmatrix} \begin{pmatrix} \mathbf{T}_-^{(1)1} & \mathbf{T}_-^{(2)1} \\ \mathbf{T}_-^{(1)2} & \mathbf{T}_-^{(2)2} \end{pmatrix}.$$

By Proposition (2.2.10), we have

$$\begin{aligned} a(\lambda) &= \mathbf{T}_-^{(1)1}\mathbf{T}_+^{(2)2} - \mathbf{T}_-^{(1)2}\mathbf{T}_+^{(2)1} = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)),\\ b(\lambda) &= \mathbf{T}_+^{(1)1}\mathbf{T}_-^{(1)2} - \mathbf{T}_+^{(1)2}\mathbf{T}_-^{(1)1} = \det(\mathbf{T}_+^{(1)}(x,\lambda), \mathbf{T}_-^{(1)}(x,\lambda)).\end{aligned}$$

□

Proposition 2.2.12. *The transition coefficient $a(\lambda)$ has an analytic extension into the upper half plane* $\mathrm{Im}\lambda \geq 0$*, with the asymptotic behavior*

$$a(\lambda) \to 1 \quad \textit{as } |\lambda| \to \infty.$$

Proof. By Lemma (2.2.11),

$$a(\lambda) = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)).$$

Then, $a(\lambda)$ has an analytic extension, since by Proposition (2.2.9), the column vectors $\mathbf{T}_-^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$ can be analytically extended into the upper half plane. □

Corollary 2.2.13. *The complex valued function $a^*(\lambda) := \overline{a}(\overline{\lambda})$ has an analytic continuation into the lower half plane* $\mathrm{Im}\lambda \leq 0$.

Proof. Suppose that

$$a(\lambda) = u(\lambda) + iv(\lambda), \quad \lambda = x + iy.$$

Taking into account that $a(\lambda)$ is analytic into the upper half plane, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

for all $y \geq 0$. Let

$$a^*(\lambda) = \hat{u}(\lambda) + i\hat{v}(\lambda).$$

Then

$$\begin{aligned} \hat{u}(x+iy) &= u(x-iy) \\ \hat{v}(x+iy) &= -v(x-iy) \end{aligned}$$

and

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial \hat{v}}{\partial y},$$

only if $y \leq 0$. Therefore, $a^*(\lambda) := \overline{a}(\overline{\lambda})$ has an analytic extension into the lower half plane, in general. □

Remark 1. *The analytic properties of* $\mathbf{T}_{\pm}^{(1)}(x,\lambda)$ *together with (*2.2.61*) imply that,* $b(\lambda)$ *has no analytic continuation off the real line.*

Now, let us analyze the zeros of the function $a(\lambda)$ on its domain of analyticity

Proposition 2.2.14. *The transition coefficient* $a(\lambda)$ *has no zeros on the real line.*

Proof. Suppose that for some real λ_0, $a(\lambda_0) = 0$. Then, by virtue of the normalization relation

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1,$$

for $\lambda = \lambda_0$, we have

$$|b(\lambda_0)|^2 = -1.$$

This is clearly a contradiction. □

For each real λ the spectral problem (2.2.55) has solution of multiplicity two. This follows from the fact that for each λ the two linearly independent columns that form the matrix $\mathbf{T}_+(x,\lambda)$ satisfy (2.2.4),(2.2.5), which is equivalent to the spectral problem (2.2.55).

Proposition 2.2.15. *If* κ *is positive, then the coefficient* $a(\lambda)$ *has no zeros on its domain of analyticity.*

Proof. Suppose that $a(\lambda_0) = 0$ with $\mathrm{Im}\lambda_0 > 0$. By Lemma 2.2.11,

$$a(\lambda) = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)). \tag{2.2.62}$$

Then, it follows that the column vectors $\mathbf{T}_-^{(1)}(x,\lambda_0)$ and $\mathbf{T}_+^{(2)}(x,\lambda_0)$ are linearly dependent. Thus, for $\lambda = \lambda_0$, the linear problem (2.2.4) has a vector column solution decaying exponentially as $|x| \to \infty$. However, that problem is equivalent to the spectral problem (2.2.55) which has a non-real eigenvalue λ_0. But as the operator $\mathscr{L}$ is self-adjoint, it only has real eigenvalues. This contradiction shows that $a(\lambda)$ has no complex zeros. □

The situation is very different from the case when $\kappa < 0$. Since $\mathscr{L}$ is not self-adjoint $a(\lambda)$ may have zeros. The analyticity and the asymptotic behavior of the coefficient $a(\lambda)$ (Proposition 2.2.12) imply that the zeros are located in a bounded region of the half plane $\mathrm{Im}\lambda \geq 0$ and may only accumulate towards the real line.

In order to simplify our analysis, we make the following assumptions:

1. No zeros occur on the real axis;

2. All the zeroes are simple.

By compactness, under the above hypotheses, it follows that $a(\lambda)$ has only a finite number of zeros and there is a strict inequality for $|b(\lambda)| < 1$.

Remark 2. *The set of functions $\psi(x)$, and $\overline{\psi}(x)$ satisfying the conditions 1 and 2, is in a natural sense, open and dense in the space of decreasing functions.*

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the complete list of zeros of $a(\lambda)$, with $\mathrm{Im}\lambda_j > 0$, $j = 1, 2, ..., n$. From the Lemma (2.2.11), we have that the column of $\mathbf{T}_-^{(1)}(x, \lambda)$ is proportional to $\mathbf{T}_+^{(2)}(x, \lambda)$. Let γ_j be the proportionality coefficient,

$$\mathbf{T}_-^{(1)}(x, \lambda_j) = \gamma_j \mathbf{T}_+^{(2)}(x, \lambda_j), \quad j = 1, 2, ..., n. \tag{2.2.63}$$

Note that

$$\gamma_j \neq 0, \tag{2.2.64}$$

because if this do not occur it implies that monodromy matrix has determinant equal to zero.

Using the involution property, we find that

$$\mathbf{T}_-^{(1)}(x, \overline{\lambda}_j) = \overline{\gamma}_j \mathbf{T}_+^{(2)}(x, \overline{\lambda}_j)$$

The set of complex numbers γ_j is one of the characteristics of the auxiliary linear problem and will play an important role in what follows. Indeed the numbers $\lambda_j, \overline{\lambda}_j$ constitute the discrete part of the spectrum of the differential operator $\mathscr{L}$ for $\kappa < 0$. Furthermore, for any κ, $\mathscr{L}$ has continuous spectrum of multiplicity two on the whole real line, according with the existence of two linearly independent solution of the spectral problem which implies the existence of the fundamental solution of the linear problem. According to observations below, we shall call $a(\lambda)$ and $b(\lambda)$ the transitions coefficients for the **continuous spectrum**, and $\gamma_j, \overline{\gamma}_j$, $j = 1, 2, ..., n$: will be called transition coefficients for the **discrete spectrum**.

Definition 2.2.2. *The set $\{b(\lambda), \lambda_j, \gamma_j\}$ is called the spectral data of the linear problem (2.2.4) if conditions*

- $\mathrm{Im}\lambda_j > 0$,
- *λ_j are simple zeroes*

hold.

Now, let us show how the analyticity of $a(\lambda)$ and the normalized relation are using to express $a(\lambda)$ through its zeros and $b(\lambda 0)$. First recall some facts from the theory of analytic functions.

Lemma 2.2.16. *Suppose we are given a complex-valued function $g(\lambda)$ which is*

- *analytic for* $\mathrm{Im}\lambda > 0$,
- *continuos for* $\mathrm{Im}\lambda > 0$,
- *vanishing at the infinity*

$$\lim_{|\lambda|\to\infty} g(\lambda) = 0. \tag{2.2.65}$$

Then, the real $\mathrm{Re}g(\lambda)$ *and imaginary* $\mathrm{Im}g(\lambda)$ *parts are related by the formula*

$$\mathrm{Im}g(\lambda) = -\frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{Re}g(\lambda)}{\mu - \lambda}\mathrm{d}\mu, \quad \lambda \in \mathbb{R} \tag{2.2.66}$$

Proof. By the Cauchy formula we have

$$g(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - \lambda}\mathrm{d}\mu, \ \mathrm{Im}\lambda > 0. \tag{2.2.67}$$

The Sokhotskii-Plemlj formula implies that

$$\begin{aligned} g(z+0) &\stackrel{\mathrm{def}}{=} \lim_{\lambda\to z} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - \lambda}\mathrm{d}\mu \\ &= \frac{g(z)}{2} + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu \end{aligned} \tag{2.2.68}$$

for every $z \in \mathbb{R}$. Here

$$\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{z-\epsilon} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu + \int_{z+\epsilon}^{\infty} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu\right) \tag{2.2.69}$$

is the principal value integral. On other hand, since $g(\lambda)$ is continuous, we have

$$g(z) = g(z+0) = \frac{g(z)}{2} + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu. \tag{2.2.70}$$

This implies

$$g(z) = \frac{1}{\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{g(\mu)}{\mu - z}\mathrm{d}\mu \tag{2.2.71}$$

□

As we already known the transition coefficients $a(\lambda)$, $b(\lambda)$ satisfy the normalization condition

$$|a(\lambda)|^2 - \varepsilon|b(\lambda)|^2 = 1 \quad \varepsilon = \mathrm{sign}\kappa. \tag{2.2.72}$$

Case $\kappa < 0$**.** The function $a(\lambda)$ is analytic in the upper half plane, continuous for $\mathrm{Im}\lambda \geq 0$ and has the asymptotic behavior

$$a(\lambda) = 1 + O(1) \quad \text{as} \ \ |\lambda| \to \infty. \tag{2.2.73}$$

Moreover, $a(\lambda)$ has zeroes $\lambda_1, ..., \lambda_n$ in the upper half plane. Let us introduce the function

$$\widetilde{a}(\lambda) = \prod_{i=1}^{n} \frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \cdot a(\lambda) \tag{2.2.74}$$

The function $\widetilde{a}(\lambda)$ is also analytic and $\widetilde{a}(\lambda) \neq 0$ for $\mathrm{Im}\lambda > 0$. It is clear that

$$|\widetilde{a}(\lambda)|^2 = |a(\lambda)|^2 = 1 - |b(\lambda)|^2, \quad \lambda \in \mathbb{R}. \tag{2.2.75}$$

The function

$$g(\lambda) \stackrel{\text{def}}{=} \ln \widetilde{a}(\lambda) = \ln |\widetilde{a}(\lambda)| + i arg \widetilde{a}(\lambda). \tag{2.2.76}$$

satisfies the hypotheses of the Lemma (2.2.16). Taking into account that

$$\mathrm{Re} g(\lambda) = \ln |\widetilde{a}(\lambda)| = \frac{1}{2} \ln(1 - |b(\lambda)|^2), \tag{2.2.77}$$

we deduce from (2.2.66) the following representation

$$\mathrm{Im} g(\lambda) = \arg \widetilde{a}(\lambda) = -\frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu, \tag{2.2.78}$$

for $\lambda \in \mathbb{R}$. We have proved the following result.

Proposition 2.2.17. *For every $\lambda \in \mathbb{R}$, we have*

$$a(\lambda) = \left(\prod_{i=1}^{n} \frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \right) \cdot (1 - |b(\lambda)|^2)^{\frac{1}{2}} \cdot \exp \left(\frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu \right) \tag{2.2.79}$$

One can rewrite (2.2.79), using the Sokhotski type formula:

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda} \mathrm{d}\mu = \pi i f(\mu) - \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - 0i} \mathrm{d}\mu. \tag{2.2.80}$$

Here $f(\mu) \to 0$ as $|\mu| \to \infty$ and

$$\int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - 0i} \mathrm{d}\mu = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - \epsilon i} \mathrm{d}\mu. \tag{2.2.81}$$

Applying (2.2.68) for $f(\mu) = \ln(1 - |b(\lambda)|^2)$, we get

$$a(\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \right) \cdot \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda - 0i} \mathrm{d}\mu \right). \tag{2.2.82}$$

One can show that this formula is valid for $\mathrm{Im}\lambda \geq 0$.

Case $\kappa > 0$. In this case, $a(\lambda)$ is analytic in the upper half plane and has no zeroes. Applying the same arguments as in the previous case to $g(\lambda) = \ln a(\lambda)$, we get

$$a(\lambda) = \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 + |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu \right). \tag{2.2.83}$$

Time evolution of spectral data.

Recall that the evolution equation (see Section 1.4), for the monodromy matrix is given by

$$\frac{\partial}{\partial t} \mathbf{T}(x, y, \lambda) = \mathbf{V}(x, \lambda) \mathbf{T}(x, y, \lambda) - \mathbf{T}(x, y, \lambda) \mathbf{V}(y, \lambda).$$

Let us consider a matrix $\mathbf{V}(x,\lambda)$ given by

$$\mathbf{V}(x,\lambda)=\mathbf{V}_0(x,\lambda)+\frac{i\lambda^2}{2}\sigma_3 \tag{2.2.84}$$

where $\mathbf{V}_0(x,\lambda)\in L_1^{2\times 2}(\mathbb{R})$ and λ is a real parameter.

Taking limits as $x\to\infty$, $y\to\pm\infty$ and multiplying by $\mathbf{E}(y,\lambda)$ on the right, we get

$$\mathbf{V}(x,\lambda)\to\mathbf{V}(\lambda)=\frac{i\lambda^2}{2}\sigma_3,\quad \text{as}\quad |x|\to\infty$$

and

$$\frac{\partial}{\partial t}\mathbf{T}_\pm(x,\lambda)=\mathbf{V}(x,\lambda)\mathbf{T}_\pm(x,\lambda)-\frac{i\lambda^2}{2}\mathbf{T}_\pm(x,\lambda)\sigma_3.$$

Performing the same operation with respect to x, we obtain the equation for the monodromy matrix

$$\frac{\partial}{\partial t}\mathbf{M}(\lambda,t)=\frac{i\lambda^2}{2}\left[\sigma_3,\mathbf{M}(\lambda,t)\right].$$

This equation posses a remarkable property: the dependence on $\psi(x)$, $\overline{\psi}(x)$ is completely eliminated. In terms of the transition coefficients for the continuous spectrum, the last equation is equivalent to

$$\frac{\partial}{\partial t}a(\lambda,t) = 0 \tag{2.2.85}$$

$$\frac{\partial}{\partial t}b(\lambda,t) = -i\lambda^2 b(\lambda,t). \tag{2.2.86}$$

In particular, we deduce that $a(\lambda)$ is time independent for real λ.

$$a(\lambda,t)=a(\lambda,0).$$

By the analyticity property, the same holds for $\mathrm{Im}\lambda>0$, so that the zeros λ_j are time independent as well. Thus in the decreasing case the generating function for the conservation laws is just $a(\lambda)$.

Let us now determine the evolution of the transition coefficients for the discrete spectrum. We have for the column vectors $\mathbf{T}_-^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$,

$$\frac{\partial}{\partial t}\mathbf{T}_-^{(1)}(x,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}_-^{(1)}(x,\lambda)-\frac{i\lambda^2}{2}\mathbf{T}_-^{(1)}(x,\lambda), \tag{2.2.87}$$

$$\frac{\partial}{\partial t}\mathbf{T}_+^{(2)}(x,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}_+^{(2)}(x,\lambda)-\frac{i\lambda^2}{2}\mathbf{T}_+^{(2)}(x,\lambda). \tag{2.2.88}$$

These relations also hold for $\mathrm{Im}\lambda>0$. They are compatible with

$$\mathbf{T}_-^{(1)}(x,\lambda_j)=\gamma_j\mathbf{T}_+^2(x,\lambda_j),\quad j=1,2,...,n, \tag{2.2.89}$$

only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j(t)=-i\lambda_j^2\gamma_j(t). \tag{2.2.90}$$

The last equation and the differential equation for $a(\lambda)$ and $b(\lambda)$ can easily be solved so that the time dependence of transition coefficients is given by the simple formulae

$$b(\lambda,t) = \mathrm{e}^{-i\lambda^2 t}b(\lambda,0), \tag{2.2.91}$$

$$\gamma_j(t) = \mathrm{e}^{-i\lambda^2 t}\gamma_j(0).\quad j=1,2,...,n. \tag{2.2.92}$$

Chapter 3

The Inverse Problem: The Rapidly Decreasing Case. The Nonlinear Schrödinger Equation

In this Chapter, we formulate results on the inverse problem for linear systems on $\mathfrak{sl}(2,\mathbb{C})$ in the rapidly decreasing case and zero curvature equation. We mean that we shall show that it is possible reconstruct the linear system in $\mathfrak{sl}(2,\mathbb{C})$ (Chapter 2) from its spectral data (definition (2.2.2)) and assuming the linear time dynamics for the spectral data (2.2.91),(2.2.92) we shall get solution for the zero curvature equation.

3.1 Formulation of results

Recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all functions $f \in C^\infty(\mathbb{R})$ such that for each integer $n \geq 0$

$$|x^m| \left| \frac{\mathrm{d}^n f}{\mathbf{d}x^n} \right| \to 0 \quad \text{as} \quad |x| \to \ \infty, \tag{3.1.1}$$

for any $m > 0$. It is clear that for every $f \in \mathcal{S}(\mathbb{R})$, we have $\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \in L_1(\mathbb{R})$. Given a $\psi \in \mathcal{S}(\mathbb{R})$, we consider the follow linear problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T}(x,y,\lambda) \ = \ \left(\mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3\right)\mathbf{T}(x,y,\lambda), \tag{3.1.2}$$

$$\mathbf{T}(x,x,\lambda) \ = \ \mathbf{I}, \tag{3.1.3}$$

where

$$\mathbf{U}_0(x) = \sqrt{\kappa}(\psi(x)\sigma_+ + \overline{\psi}(x)). \tag{3.1.4}$$

We shall refer to the Cauchy problem (3.1.2),(3.1.3) as *the linear problem corresponding to ψ in the rapidly decreasing case.*

The form of the coefficient $\mathbf{U}(x,\lambda)$ in (3.1.2) implies that involution property holds (Proposition 2.1.3) and hence, its monodromy matrix $\mathbf{M}(\lambda)$ (2.2.49) can be written

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix},$$

where $\epsilon = \mathrm{sign}(\kappa)$ (Proposition 2.1.5). In Subsection 2.2.3, we show that in the decreasing case the linear system (3.1.2) is interpreted as the spectral problem:

$$\mathscr{L}\mathbf{f} = \frac{\lambda}{2}\mathbf{f},$$

where $\mathscr{L}$ is the differential linear operator given by (2.2.55). This gives rise the notion of spectral data for the linear problem (3.1.2),(3.1.3) which consist of the function $b(\lambda)$ for $\kappa > 0$, and the set $\{(b(\lambda), \overline{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ for $\kappa < 0$. Follow [7], by inverse problem we mean the reconstruction of the matrix coefficients of $\mathbf{U}(x,\lambda)$ in (3.1.2) from spectral data. We have two main results for the inverse problem, which depend of the sign of the constant κ in the matrix coefficient $\mathbf{U}(x,\lambda)$ (3.1.4).

Theorem 3.1.1 (Case $\kappa > 0$)**.** *Let $b(\lambda) \in \mathcal{S}(\mathbb{R})$ be a complex valued function satisfying*

$$|b(\lambda)| < 1.$$

Define

$$a(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1+|b(\mu)|)}{\mu-\lambda-i0}\mathrm{d}\mu\right\}. \tag{3.1.5}$$

Then there exist an unique $\psi(x) \in \mathcal{S}(\mathbb{R})$ such that

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}$$

*is the monodromy matrix of the linear problem (*3.1.2*) corresponding to $\psi(x)$; and*

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1, \quad \textit{for all real } \lambda; \tag{3.1.6}$$

Theorem 3.1.2 (Case $\kappa < 0$)**.** *Let $\{(b(\lambda), \overline{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ be a set consisting of complex number λ_j, γ_j ($j = 1, 2, ..., n$) and $b(\lambda) \in \mathcal{S}(\mathbb{R})$ which satisfy the following condition:*

- $\mathrm{Im}\lambda_j > 0$, $\lambda_i \neq \lambda_j$ *if* $i \neq j$,
- $\gamma_j \neq 0$,
- $|b(\lambda)| < 1$.

Define

$$a(\lambda) = \prod_{j=1}^{n}\frac{\lambda-\lambda_j}{\lambda-\overline{\lambda}_j}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1+\epsilon|b(\mu)|)}{\mu-\lambda-i0}\mathrm{d}\mu\right\}. \tag{3.1.7}$$

Then there exist an unique $\psi(x) \in \mathcal{S}(\mathbb{R})$ such that

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix}$$

*is the monodromy matrix of the linear problem (*3.1.2*) corresponding to $\psi(x)$, satisfying:*

1. $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$, *for all real* λ;
2. *the columns* $\left(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)\right)$ *of the Jost solutions* $\mathbf{T}_\pm(x,\lambda)$ *in (*2.2.46*)-(*2.2.47*) satisfy the conditions*

$$\mathbf{T}_-^{(1)}(x,\lambda_j) = \gamma_j\mathbf{T}_+^{(2)}(x,\lambda_j), \quad j = 1, 2, ..., n. \tag{3.1.8}$$

The basic tool for solving the inverse problem formulated in the proceed Theorems is provided by the **Riemann-Hilbert Problem** or analytic factorization problem [7].

The theorem above can be interpreted in terms of differential operators and spectral data. For a given set $\{(b(\lambda), \bar{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ satisfying the conditions of Theorem (3.1.2), there exist a unique complex valued function $\psi \in \mathcal{S}(\mathbb{R})$ such that the corresponding linear problem this set as spectral data.

Sketch proof of Theorem (3.1.1) and Theorem (3.1.2). The Theorems formulated above formally establish the existence of the linear problem for the given spectral data. However, they do not say how that linear problem can be obtained. The motivation for analyze the proof of these Theorems is mainly to give a complete study of this result and secondly is because it provides essentially how to derive the linear system in question. There is no difference in the proof of both theorems up in the application of the Riemann Problem.

We give the scheme of the proof in steps.

Step one. Let us assume that the complex valued function $b(\lambda)$ satisfy the condition of Theorem (3.1.1) or (3.1.2). We define the matrix valued function $\mathbb{R}^2 \ni (x, y) \mapsto \mathbf{G}(x, \lambda)$

$$\mathbf{G}(x,\lambda) \stackrel{\text{def}}{=} \mathbf{E}(x,\lambda)\mathbf{G}(\lambda)\mathbf{E}^{-1}(x,\lambda) = \begin{pmatrix} 1 & \epsilon\bar{b}(\lambda)\mathrm{e}^{-i\lambda x} \\ -b(\lambda)\mathrm{e}^{i\lambda x} & 1 \end{pmatrix}, \tag{3.1.9}$$

with

$$\mathbf{E}(x,\lambda) = \exp\left\{\frac{x\lambda}{2i}\sigma_3\right\}, \quad \mathbf{G}(x,\lambda) = \begin{pmatrix} 1 & \epsilon\bar{b}(\lambda) \\ -b(\lambda) & 1 \end{pmatrix}. \tag{3.1.10}$$

The matrix $\mathbf{G}(x, \lambda)$ has the following properties:

1. $\det \mathbf{G}(x,\lambda) = 1 + \epsilon|b(\lambda)|^2 > 0$, because $|b(\lambda)| < 1$.
2. Since $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ we have $\mathbf{G}(x,\lambda) = \mathbf{I} + o(1)$, as $|\lambda| \to \infty$
3. Integral representation

$$\mathbf{G}(x,\lambda) = \mathbf{I} + \int_{-\infty}^{\infty} \Phi(x+s)\mathrm{e}^{i\lambda s}\mathrm{d}s, \tag{3.1.11}$$

where

$$\Phi(s) = \begin{pmatrix} 0 & \epsilon\overline{\beta}(-s) \\ -\beta(s) & 0 \end{pmatrix}, \tag{3.1.12}$$

and $\beta(s)$ is given by

$$\beta(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\lambda)\mathrm{e}^{-i\lambda s}\mathrm{d}\lambda. \tag{3.1.13}$$

By the Riemman Problem (see Appendix) there exit unique matrices $\mathbf{G}_\pm(x, \lambda)$ such that

- $\mathbf{G}_+(x, \lambda)$ is analytic into the upper half plane for $\mathrm{Im}\lambda \geq 0$.
- $\mathbf{G}_-(x, \lambda)$ is analytic into the lower half plane for $\mathrm{Im}\lambda \leq 0$.
- For each $\lambda \in (-\infty, \infty)$,

$$\mathbf{G}(x,\lambda) = \mathbf{G}_+(\lambda)\mathbf{G}_-(\lambda). \tag{3.1.14}$$

Step two. Let us define the matrices

$$\mathbf{T}_+(x,\lambda) = \mathbf{G}_+^{-1}(x,\lambda)\mathbf{E}(x,\lambda) \tag{3.1.15}$$

and

$$\mathbf{T}_-(x,\lambda) = \mathbf{G}_-(x,\lambda)\mathbf{E}(x,\lambda). \tag{3.1.16}$$

We claim that $\mathbf{T}_\pm(x,\lambda)$ satisfy the linear problem

$$\frac{\mathrm{d}\mathbf{T}_\pm}{\mathrm{d}x} = \left(\frac{\lambda}{2i}\sigma_3 + \mathbf{U}_0(x)\right)\mathbf{T}_\pm, \tag{3.1.17}$$

where $\mathbf{U}_0(x)$ is in $L_1^{n\times n}(\mathbb{R}_x)$. We have the following facts about $T_\pm$ which are consequence of their definition (3.1.15), (3.1.16),

- $\mathbf{T}_+$ is analytic and non-degenerate in the upper half plane, except, maybe, for simple poles at $\lambda = \lambda_j$, $j = 1, 2, .., n$.
- $\mathbf{T}_-$ is analytic and non-degenerate in the lower half plane, except, maybe, for simple poles at $\lambda = \overline{\lambda}_j$, $j = 1, 2, .., n$.
- $$\mathbf{T}_-(x,\lambda) = \mathbf{T}_+(x,\lambda)\mathbf{G}(\lambda). \tag{3.1.18}$$

Differentiating (3.1.18) with respect to x we get

$$\frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x,\lambda)\mathbf{T}_+^{-1}(x,\lambda) = \frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x,\lambda)\mathbf{T}_-^{-1}(x,\lambda) \tag{3.1.19}$$

We claim, without proof [7], that both sides of (3.1.19) are entire functions on λ and we shall analyze its asymptotic behavior as $|\lambda| \to \infty$. For this purpose, we realize that since $\mathbf{G}_-(x,\lambda)$ is analytic in the lower half plane the function $\mathbf{T}_-$ has the representation

$$\mathbf{T}_-(x,\lambda) = \left(\mathbf{I} + \int_0^\infty \Phi_-(x+s)\mathrm{e}^{-i\lambda s \mathrm{d}s}\right)\mathbf{E}(x,\lambda), \tag{3.1.20}$$

where $\Phi_-(x+s)$ is absolutely continuous in x, and $\frac{\partial\Phi_-}{\partial x}$, $\frac{\partial\Phi_-}{\partial s}$, $\frac{\partial^2\Phi_-}{\partial x\partial s}$ as function of s belong to $L_1^{2\times 2}(0,\infty)$ [7]. Then, for $\mathrm{Im}\lambda \le 0$ has the asymptotic behavior

$$\mathbf{T}_-(x,\lambda) = \left(\mathbf{I} + \frac{\Phi_-(x,0)}{i\lambda} + o\left(\frac{1}{|\lambda|}\right)\right)\mathbf{E}(x,\lambda), \quad \text{as } |\lambda| \to \infty. \tag{3.1.21}$$

It follows that

$$\frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x,\lambda)\mathbf{T}_-^{-1}(x,\lambda) = \frac{\lambda\sigma_3}{2i} + \frac{1}{2}\left[\sigma_3, \Phi_-(x,0)\right] + o(1)\text{as } |\lambda| \to \infty. \tag{3.1.22}$$

Similarly, we obtain from

$$\mathbf{T}_+(x,\lambda) = \mathbf{E}^{-1}(x,\lambda)\left(\mathbf{I} + \int_0^\infty \Phi_+(x+s)\mathrm{e}^{i\lambda s \mathrm{d}s}\right), \tag{3.1.23}$$

that for $\mathrm{Im}\lambda \ge 0$,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x,\lambda)\mathbf{T}_+^{-1}(x,\lambda) &= \mathbf{T}_+(x,\lambda)\frac{\mathrm{d}\mathbf{T}_+^{-1}}{\mathrm{d}x}(x,\lambda) \\ &= \frac{\lambda\sigma_3}{2i} + \frac{1}{2}\left[\sigma_3, \Phi_+(x,0)\right] + o(1)\text{as } |\lambda| \to \infty. \end{aligned}$$

Hence, by the Liouville Theorem we get

$$\begin{aligned}\frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x,\lambda)\mathbf{T}_+^{-1}(x,\lambda) &= \frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x,\lambda)\mathbf{T}_-^{-1}(x,\lambda)\\ &= \frac{\lambda\sigma_3}{2i}+\mathbf{U}_0(x),\end{aligned} \tag{3.1.24}$$

where

$$\mathbf{U}_0(x)=\frac{1}{2}\left[\sigma_3,\Phi_+(x,0)\right]=\frac{1}{2}\left[\sigma_3,\Phi_-(x,0)\right]. \tag{3.1.25}$$

To conclude we must note that $\mathbf{U}_0$ is anti-diagonal and satisfy involution property, therefore it can be written

$$\mathbf{U}_0=\sqrt{\kappa}\begin{pmatrix}0 & \overline{\psi}(x)\\ \psi(x) & 0\end{pmatrix}. \tag{3.1.26}$$

Step Three To conclude that

$$\mathbf{M}(\lambda)=\begin{pmatrix}a(\lambda) & \epsilon\bar{b}(\overline{\lambda})\\ b(\lambda) & \overline{a}(\overline{\lambda})\end{pmatrix}$$

is the monodromy matrix for the linear system corresponding to $\psi(x)$. For this purpose, follow [7] we have analyze the asymptotic behavior for $\mathbf{G}_\pm(x,\lambda)$ and realize that they are composed by the Jost solution of the linear problem for Ψ, which defined de monodromy matrix (Chapter 2).

Formulae for ψ.

Case $\kappa>0$. Let

$$\beta(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(\lambda)\mathrm{e}^{-i\lambda s}d\lambda \tag{3.1.27}$$

be the Fourier transform of $b(s)\in\mathcal{S}(\mathbb{R})$. Then $\beta\in\mathcal{S}(\mathbb{R})$ Let $C^+(x,s)$ be the solution of the Winner-Hopf integral equation

$$C^+(x,s)=\beta(s-x)-\int_0^\infty\left(\int_x^\infty\beta(s-t)\overline{\beta}(s'-t)\mathrm{d}t\right)\cdot C^+(x,s')ds' \tag{3.1.28}$$

for $s\geq 0$. Then,

$$\psi(x)=\frac{1}{\sqrt{\kappa}}C^+(x,0). \tag{3.1.29}$$

Case $\kappa<0$. In this case $\psi(x)$ consists of two parts $\psi(x)=\psi^c(x)+\psi^d(x)$.

The first part ψ^c is defined by $b(\lambda)$,

$$\psi^c(x)=\frac{1}{\sqrt{\kappa}}C^-(x,0) \tag{3.1.30}$$

where $C^-(x,0)$ is the solution of the Winer-Hopf equation

$$C^-(x,s)=\beta(s-x)-\int_0^\infty\left(\int_x^\infty\beta(s-t)\overline{\beta}(s'-t)\mathrm{d}t\right)\cdot C^-(x,s')ds' \tag{3.1.31}$$

The second part ψ^d is determined by the data $(\lambda_i,\gamma_i),b(\lambda)$ in the context of the Riemann problem. In general, when $b\neq 0$ the problem is reduce to the computing the Blashke factor and is quit complicated.

Now, we consider the particular case when

$$b(\lambda)\equiv 0. \tag{3.1.32}$$

In this case, the function $a(\lambda)$ take the form

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j}, \tag{3.1.33}$$

and the inverse problem can be solved in closed form.

If $\kappa > 0$, then $\beta \equiv 0$ and hence, $\psi \equiv 0$.

Now, let us assume $\kappa < 0$. Let $\{b(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma}_j\}$ be the spectral data, then the function $\psi \in \mathcal{S}(\mathbb{R})$ is given by

$$\psi(x) = \frac{i}{\sqrt{\kappa}} \sum_{j=1}^{n} \overline{p}_j(x), \tag{3.1.34}$$

where, coefficients $p_j(x)$ satisfy the linear system of equations

$$\sum_{k=1}^{n} \left(\frac{1 + \overline{\gamma}_j(x)\gamma_k(x)}{\overline{\lambda}_j - \lambda_k} \right) p_k(x) = \overline{\gamma}_j(x), \quad j = 1, 2, ..., n; \tag{3.1.35}$$

with

$$\gamma_j(x) = \gamma_j \mathrm{e}^{i\lambda_j x}. \tag{3.1.36}$$

In order to illustrate the formula (3.1.34), we derive the case $n = 1$ and $n = 2$.

Taking $n = 1$ in (3.1.35), we get

$$\frac{1 + |\gamma_1(x)|^2}{-2i\mathrm{Im}(\lambda_1)} p_1(x) = \overline{\gamma}_1(x). \tag{3.1.37}$$

Solving the last equation for p_1 and substituting in (3.1.34) we obtain

$$\psi(x) = \frac{2\mathrm{Im}\lambda_1}{\sqrt{\kappa}} \cdot \frac{\gamma_1(x)}{1 + |\gamma_1(x)|^2}, \tag{3.1.38}$$

where

$$\gamma_1(x) = \gamma_1 \mathrm{e}^{i\lambda_1 x}. \tag{3.1.39}$$

For $n = 2$ we get the following algebraic system

$$\begin{aligned} \left(\frac{1 + \overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1 - \lambda_1} \right) p_1(x) + \left(\frac{1 + \overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1 - \lambda_2} \right) p_2(x) &= \overline{\gamma}_1(x), \\ \left(\frac{1 + \overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2 - \lambda_1} \right) p_1(x) + \left(\frac{1 + \overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2 - \lambda_2} \right) p_2(x) &= \overline{\gamma}_2(x). \end{aligned}$$

Solving for the unknown functions $p_1(x)$ and $p_2(x)$,

$$p_1(x) = \frac{\overline{\gamma}_1(x) \frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2 - \lambda_2} - \overline{\gamma}_2(x) \frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1 - \lambda_2}}{\left(\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1 - \lambda_1}\right)\left(\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2 - \lambda_2}\right) - \left(\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2 - \lambda_1}\right)\left(\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1 - \lambda_2}\right)}, \tag{3.1.40}$$

$$p_2(x) = \frac{\overline{\gamma}_1(x) \frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2 - \lambda_1} - \overline{\gamma}_2(x) \frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1 - \lambda_1}}{\left(\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1 - \lambda_1}\right)\left(\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2 - \lambda_2}\right) - \left(\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2 - \lambda_1}\right)\left(\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1 - \lambda_2}\right)}, \tag{3.1.41}$$

and

$$\psi(x) = i \frac{\overline{p}_1 + \overline{p}_2}{\sqrt{\kappa}}. \tag{3.1.42}$$

3.2 The inverse Problem for Zero curvature Equation

We shall formulate the result on the reconstruction of solutions of the zero curvature equation on $\mathfrak{sl}(2\mathbb{C})$,

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}] = 0, \tag{3.2.1}$$

from spectral data which appear in Theorems (3.1.2),(3.1.1).

The following results are a direct consequence of Theorems (3.1.2),(3.1.1).

Proposition 3.2.1 (Case $\kappa > 0$)**.** *Let $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ be a complex valued function satisfying*

$$|b(\lambda)| < 1.$$

Let us define

$$\begin{aligned} b(\lambda, t) &= \mathrm{e}^{-i\lambda^2 t} b(\lambda), \qquad (3.2.2)\\ a(\lambda, t) &= \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 + |b(\mu, t)|)}{\mu - \lambda - i0} \mathrm{d}\mu \right\}. \qquad (3.2.3) \end{aligned}$$

for $t \in \mathbb{R}$. Then, for each $t \in \mathbb{R}$

$$|a(\lambda, t)|^2 + |b(\lambda, t)|^2 = 1, \quad \textit{for all real } \lambda; \tag{3.2.4}$$

and there exist an unique $\psi(x, t) \in \mathcal{S}(\mathbb{R}_x)$ such that

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda, t) & \bar{b}(\lambda, t) \\ b(\lambda, t) & \bar{a}(\lambda, t) \end{pmatrix}$$

is the monodromy matrix of the linear problem (2.2.4) corresponding to $\psi(x, t)$.

Proof. For each fixed $t \in \mathbb{R}$ we have

$$\begin{aligned} |b(\lambda, t)|^2 &= b(\lambda, t)\bar{b}(\lambda, t) \\ &= \mathrm{e}^{-i\lambda^2 t} b(\lambda) \mathrm{e}^{i\lambda^2 t} \bar{b}(\lambda) \\ &= \mathrm{e}^{-i\lambda^2 t} \mathrm{e}^{i\lambda^2 t} b(\lambda)\bar{b}(\lambda) = b(\lambda)\bar{b}(\lambda) \\ &= |b(\lambda)|^2 \\ &< 1. \end{aligned}$$

It is clear that $b(\lambda, t)$ belongs to $\mathcal{S}(\mathbb{R})$, because this space is a ring of functions and both $\mathrm{e}^{-i\lambda^2 t}$, $b(\lambda)$ are in $\mathcal{S}(\mathbb{R})$. It follows from Theorem (3.1.1) the existence of the function $\psi(x, t)$ which define a liner problem of type (2.2.4) for each $t \in \mathbb{R}$ and we get the rest of the proposition. □

Proposition 3.2.2 (Case $\kappa < 0$)**.** *Let $\{(b(\lambda), \bar{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ be a set consisting of complex number λ_j, γ_j $(j = 1, 2, ..., n)$ and $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ which satisfy the following condition:*

- $\mathrm{Im}\lambda_j > 0$, $\lambda_i \neq \lambda_j$ *if* $i \neq j$,
- $\gamma_j \neq 0$,
- $|b(\lambda)| < 1$.

Let us define

$$b(\lambda,t) = \mathrm{e}^{-i\lambda^2 t} b(\lambda), \tag{3.2.5}$$

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda-\lambda_j}{\lambda-\overline{\lambda_j}} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1+\epsilon|b(\mu,t)|)}{\mu-\lambda-i0}\mathrm{d}\mu\right\}. \tag{3.2.6}$$

for $t\in\mathbb{R}$. *Then, for each* $t\in\mathbb{R}$

$$|a(\lambda,t)|^2+|b(\lambda,t)|^2=1, \quad \textit{for all real}\ \ \lambda; \tag{3.2.7}$$

and there exist an unique $\psi(x,t)\in\mathcal{S}(\mathbb{R}_x)$ *such that*

$$\mathbf{M}(\lambda,t)=\begin{pmatrix} a(\lambda,t) & \epsilon\bar{b}(\lambda,t)\\ b(\lambda,t) & \overline{a}(\lambda,t)\end{pmatrix}$$

*is the monodromy matrix of the linear problem (*2.2.4*) corresponding to* $\psi(x,t)$ *and the columns* $\left(\mathbf{T}_-^{(1)}(x,t,\lambda),\mathbf{T}_+^{(2)}(x,t,\lambda)\right)$ *of the Jost solutions* $\mathbf{T}_\pm(x,t,\lambda)$ *in (*2.2.46*)-(*2.2.47*) satisfy the conditions*

$$\mathbf{T}_-^{(1)}(x,t,\lambda_j)=\gamma_j(t)\mathbf{T}_+^{(2)}(x,t,\lambda_j), \quad j=1,2,...,n. \tag{3.2.8}$$

where

$$\gamma_i(t)=\mathrm{e}^{-i\lambda^2 t}\gamma_i. \tag{3.2.9}$$

The proof of this statement follows the same arguments as Proposition (3.2.1).

Now we formulate the main result

Theorem 3.2.3. *Let* $\psi(x,t)$ *be the function defined in Proposition (*3.2.1*) or in Proposition (*3.2.2*). The* $\mathfrak{sl}(2,\mathbb{C})$*-valued functions* $\mathbf{U}(x,t,\lambda)$ *and* $\mathbf{V}(x,t,\lambda)$ *defined by*

$$\mathbf{U}(x,t,\lambda) = \mathbf{U}_0(x,t)+\lambda\mathbf{U}_1 \tag{3.2.10}$$

$$\mathbf{V}(x,t,\lambda) = \mathbf{V}_0(x,t)+\lambda\mathbf{V}_1(x,t)+\lambda^2\mathbf{V}_2(x,t), \tag{3.2.11}$$

where

$$\mathbf{U}_0(x,t)=\sqrt{\kappa}\begin{pmatrix} 0 & \overline{\psi}(x,t)\\ \psi(x,t) & 0\end{pmatrix}, \quad \mathbf{U}_1=\frac{1}{2i}\sigma_3.$$

$$\mathbf{V}_1(x,t)=-\mathbf{U}_0(x,t), \quad \mathbf{V}_2(x,t)=-\mathbf{U}_1,$$

and

$$\mathbf{V}_0(x,t)=i\sigma_3\mathbf{U}_0^2(x,t)+i\frac{\partial\mathbf{U}_0(x,t)}{\partial x}\sigma_3.$$

*satisfy the zero curvature equation (*3.2.1*), for any* λ.

In the case $b=0$ we can derive explicit solution for zero curvature equation. We are going to give derivation of the corresponding formulaes in chapter four where we consider nonlinear Schrodinger equation.

Sketch of the proof of Theorem (3.2.3). Using the function $\psi(x,t)$ given in the Proposition (3.2.1) or Proposition (3.2.2) we can construct the matrix $\mathbf{U}(x,t,\lambda)$ (3.2.10). We only have to prove that the matrix $\mathbf{V}(x,t,\lambda)$ of the form (3.2.11) satisfy together with $\mathbf{U}$ the zero curvature equation.

Let us define the matrix valued function

$$\mathbf{G}(x,t\lambda)=\mathbf{E}^{-1}(t,\lambda^2)\mathbf{G}(x,t,\lambda)\mathbf{E}(t,\lambda^2), \tag{3.2.12}$$

where $\mathbf{E}(t,\lambda^2) = \exp\left\{\frac{\lambda^2 t}{2}\sigma_3\right\}$ and $\mathbf{G}(x,t,\lambda)$ given by (3.1.9).

Consider the Riemann problem

$$\mathbf{G}(x,t,\lambda) = \mathbf{G}_+(x,t,\lambda)\mathbf{G}_-(x,t,\lambda) \tag{3.2.13}$$

which is uniquely solvable in the Schwartz space for x and t. Let $\mathbf{T}_\pm(x,t,\lambda)$ be the function of the form

$$\begin{aligned}\mathbf{T}_+(x,t,\lambda) &= \mathbf{G}_+^{-1}(x,t,\lambda)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2)\\ \mathbf{T}_-(x,t,\lambda) &= \mathbf{G}_-(x,t,\lambda)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2).\end{aligned}$$

For each fixed t the functions $\mathbf{T}_\pm(x,t,\lambda)$ satisfy the linear problem corresponding to $\psi(x,t)$ (see the sketchs'proof of Theorem 3.1.1 and Theorem 3.1.2). We shall show that they also satisfy a differential equation with respect to t for each fixed x.

Since $\mathbf{G}_\pm(x,t,\lambda)$ satisfy the Riemann problem (3.2.13) we get

$$\mathbf{T}_-(x,t,\lambda) = \mathbf{T}_+(x,t,\lambda)\mathbf{G}(\lambda), \tag{3.2.14}$$

and from this relation we derive

$$\frac{\partial \mathbf{T}_-}{\partial t}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \frac{\partial \mathbf{T}_+}{\partial t}(x,t,\lambda)\mathbf{T}_+^{-1}(x,t,\lambda). \tag{3.2.15}$$

We remark that the functions $\frac{\partial \mathbf{T}_-}{\partial x}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda)$ are non-singular in their respective domain of analyticity and hence, by (3.2.15) give rise to an entire function of λ. Proceeding in the same manner as in sketchs'proof of Theorem (3.1.1) and Theorem (3.1.2), we use the integral representation

$$\mathbf{T}_-(x,t,\lambda) = \left(\mathbf{I} + \int_0^\infty \Phi_-(x,t,s)\mathrm{e}^{-i\lambda s}\mathrm{d}s\right)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2), \tag{3.2.16}$$

and derive the asymptotic expansion

$$\mathbf{T}_-(x,t,\lambda) = \left(\mathbf{I} + \frac{\Phi_-(x,t,0)}{i\lambda}) - \frac{1}{\lambda^2}\frac{\partial \Phi_-}{\partial s}(x,t,0) + O\left(\frac{1}{|\lambda|^3}\right)\right)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2), \tag{3.2.17}$$

as $|\lambda| \to \infty$. Differentiating with respect to t, we obtain

$$\frac{\partial \mathbf{T}_-}{\partial t}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \mathbf{V}(x,t,\lambda) + O\left(\frac{1}{|\lambda|}\right), \tag{3.2.18}$$

where

$$\mathbf{V}(x,t,\lambda) = \lambda^2\mathbf{V}_2 + \lambda\mathbf{V}_1 + \mathbf{V}_0, \tag{3.2.19}$$

with

$$\mathbf{V}_2(x,t) = \frac{i\sigma_3}{2}, \quad \mathbf{V}_1(x,t) = \frac{1}{2}\left[\Phi_-(x,t,0),\sigma_3\right] = -\mathbf{U}_0(x,t), \tag{3.2.20}$$

and

$$\mathbf{V}_0(x,t) = \frac{i}{2}\left[\sigma_3, \frac{\partial \Phi_-}{\partial s}(x,t,0)\right] + \frac{i}{2}\left[\Phi_-(x,t,0),\sigma_3\right]\Phi_-(x,t,0). \tag{3.2.21}$$

On other hand, applying successively integration by parts to (3.2.16) and differentiating with respect to x we find

$$\frac{\partial \mathbf{T}_-}{\partial x}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \frac{\lambda\sigma_3}{2i} + \mathbf{U}_0(x) + \sum_{n=1}^{\infty} \frac{\mathbf{T_n(x,t)}}{(i\lambda)^n} + O\left(|\lambda|^{-\infty}\right) \tag{3.2.22}$$

as $|\lambda| \to \infty$, $\mathrm{Im}\lambda < 0$. In particular, we have

$$\begin{aligned}\mathbf{T}_1(x,t) &= \frac{1}{2}\left(\left[\sigma_3, \frac{\partial \Phi_-}{\partial s}(x,t,0)\right] + \left[\Phi_-(x,t,0),\sigma_3\right]\Phi_-(x,t,0) + 2\frac{\partial \Phi_-}{\partial x}(x,t,0)\right) \\ &= -i\mathbf{V}_0(x,t) + \frac{\partial \Phi_-}{\partial x}(x,t,0).\end{aligned}$$

However, $\mathbf{T}_-$ satisfies the linear problem corresponding to $\psi(x,t)$ and it follows that

$$\mathbf{T}_n(x,t) = 0 \quad \text{for } n = 1,2,...$$

This implies that

$$\mathbf{V}_0(x,t) = -i\frac{\partial \Phi_-}{\partial x}(x,t,0). \tag{3.2.23}$$

We can deduce from equation (3.2.20) that the anti-diagonal part of $\frac{\partial \Phi_-}{\partial x}(x,t,0)$ is equal to $-\frac{\partial U_0}{\partial x}(x,t)\sigma_3$. To find the diagonal part, we consider again that $\mathbf{T}_1(x,t) = 0$ and using the equations (3.2.20) and (3.2.23) we get that it equals to $-\sigma_3\mathbf{U}^2(x,t)$. Finally we obtain

$$\mathbf{V}_0(x,t) = i\sigma_3\mathbf{U}_0^2(x,t) + i\frac{\partial \mathbf{U}_0(x,t)}{\partial x}\sigma_3.$$

Similarly, we can obtain

$$\frac{\partial \mathbf{T}_+}{\partial t}(x,t,\lambda)\mathbf{T}_+^{-1}(x,t,\lambda) = \mathbf{V}(x,t,\lambda) + O\left(\frac{1}{|\lambda|}\right). \tag{3.2.24}$$

Therefore $\mathbf{T}_\pm(x,t,\lambda)$ satisfy the linear system

$$\frac{\partial \mathbf{T}_\pm}{\partial t}(x,t,\lambda)\mathbf{V}(x,t,\lambda)\mathbf{T}_\pm(x,t,\lambda);$$

now, recalling that they also are solution of

$$\frac{\partial \mathbf{T}_\pm}{\partial x}(x,t,\lambda)\mathbf{U}(x,t,\lambda)\mathbf{T}_\pm(x,t,\lambda),$$

it follows that the two linear system are compatible ant it implies the zero curvature equation

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0.$$

3.3 The Nonlinear Schrödinger Equation

The notion of integrable ODE is related with the existence of a number of first integrals. Specially, the integrability property is clear for finite dimensional Hamiltonian systems [3]. In the infinite dimensional case, the situation is more complicated. One of the possible definitions is that [9]: a system of nonlinear differential equations is integrable if it can be represented as the consistence condition of an overdeterminated linear system which is equivalent to zero curvature equation with spectral parameter. An example of an integral system is **the Nonlinear Schrödinger equation (NLS equation)**, a dynamical system generated by the equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + 2\kappa|\psi|^2\psi \quad (\kappa \in \mathbb{R}) \tag{3.3.1}$$

with the initial condition

$$\psi(x,t)|_{t=0} = \psi(x). \tag{3.3.2}$$

As an application of the inverse problem discussed in Chapter 3, we construct some solutions of the Nonlinear Schrödinger Equation (NLS equation). This equation arises in various physical contexts, for example, it describes the effects of self-focusing of the envelope of a monochromatic plane wave propagating in nonlinear media [2]. The NLS equation appears also in the theory of surfaces waves on shallow water [4]. Equation (3.3.1) may be also considered as the Hatree-Fock equation for one dimensional quantum Boson gas equation with point intersection . Physically, the constant κ in (3.3.1) plays the role of acoupling constant: the case $\kappa > 0$ corresponds to attractive interaction and $\kappa < 0$ is the repulsive case. The two cases are essentially different in optical applications, describing self-focusing or defocusing of the light rays [2]. Mathematically, these two cases are also very different because the first one correspond to a selfadjoint linear problem while the second one is related a non-selfadjoint linear problem. The nonlinear Schrödinger equation was first solved by the inverse scattering method by Zakharov and Shabat [13]. In our treatment we shall follow an approach [7], using the result of Chapter 3. In the context of the integrability of NLS equation, the key observation is that, the NLS equation admits a zero curvature representation or Lax's pair.

3.4 Zero curvature representation for NLS equation

Consider the zero curvature equation on $\mathfrak{sl}(2,\mathbb{C})$

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0. \tag{3.4.1}$$

Suppose that $\mathbf{U}(x,t,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$ is of the form

$$\mathbf{U}(x,t,\lambda) = \mathbf{U}_0(x,t) + \frac{\lambda}{2i}\sigma_3, \quad \mathbf{U}_0(x,t) = \sqrt{\kappa}\begin{pmatrix} 0 & \overline{\Phi} \\ \Phi & 0 \end{pmatrix} \tag{3.4.2}$$

where $\Phi = \Phi(x,t)$ is a differentiable complex valued function, $\lambda \in \mathbb{C}$ is a parameter and $\kappa \in \mathbb{R}$ is a constant. Recall that representation (3.4.2) comes from a claim of λ-parameter curves in $\mathfrak{sl}(2,\mathbb{C})$ satisfying the involution property (see Chapter 2). Moreover, let us choose $\mathbf{V}(x,t,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$ as follows

$$\mathbf{V} = \mathbf{V}_0 - \lambda\mathbf{U}_0 - \frac{\lambda^2}{2i}\sigma_3, \tag{3.4.3}$$

$$\mathbf{V}_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{3.4.4}$$

where α,β and γ are some smooth functions of x and t. It is easy to see that $\mathbf{U}$ and $\mathbf{V}$ in (3.4.2) and (3.4.3) satisfy (3.4.1) for all λ if and only if

$$\frac{\partial \mathbf{U}_0}{\partial t} - \frac{\partial \mathbf{V}_0}{\partial x} + [\mathbf{U}_0, \mathbf{V}_0] = 0, \tag{3.4.5}$$

$$2i\frac{\partial \mathbf{U}_0}{\partial x} + [\sigma_3, \mathbf{V}_0] = 0. \tag{3.4.6}$$

The following observation clarify our choice of representation (3.4.3) for $\mathbf{V}$.

Proposition 3.4.1. *The differential operators*

$$\mathscr{L} = i\frac{\partial}{\partial x} - i\mathbf{U}_0 + \frac{\lambda}{2}\sigma_3 \tag{3.4.7}$$

$$\mathscr{M} = \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial x} - \mathbf{V}_0, \tag{3.4.8}$$

commute for all λ *if and only if* $\mathbf{U}_0$ *and* $\mathbf{V}_0$ *satisfy* (3.4.5)*,*(3.4.6)*.*

The proof is the straight forward computation. Thus, the zero curvature equation (3.4.1) in the class of matrix functions of the form (3.4.2),(3.4.3) is reduced to the commutativity condition

$$[\mathscr{L}, \mathscr{M}] = 0.$$

Now, we proceed to the study of equations (3.4.5),(3.4.6). Putting (3.4.2),(3.4.3) into (3.4.5),(3.4.6) we get the following set of equations for functions Φ, α, β, γ :

$$-\frac{\partial \alpha}{\partial x} + \sqrt{\kappa}\left(\overline{\Phi}\gamma - \Phi\beta\right) = 0, \tag{3.4.9}$$

$$\sqrt{\kappa}\frac{\partial \overline{\Phi}}{\partial t} - \frac{\partial \beta}{\partial x} - 2\sqrt{\kappa}\,\overline{\Phi}\alpha = 0, \tag{3.4.10}$$

$$\sqrt{\kappa}\frac{\partial \Phi}{\partial t} - \frac{\partial \gamma}{\partial x} + 2\sqrt{\kappa}\,\Phi\alpha = 0, \tag{3.4.11}$$

$$i\sqrt{\kappa}\frac{\partial \overline{\Phi}}{\partial x} + \beta = 0, \tag{3.4.12}$$

$$i\sqrt{\kappa}\frac{\partial \Phi}{\partial x} - \gamma = 0. \tag{3.4.13}$$

The last relations (3.4.12), (3.4.13) give

$$\beta = -i\sqrt{\kappa}\frac{\partial \overline{\Phi}}{\partial x}, \quad \gamma = i\sqrt{\kappa}\frac{\partial \Phi}{\partial x}. \tag{3.4.14}$$

Then, from (3.4.9) it follows that

$$\frac{\partial \alpha}{\partial x} = i\kappa\frac{\partial}{\partial x}\left(\overline{\Phi}\Phi\right) \tag{3.4.15}$$

and hence

$$\alpha = i\kappa\left(\overline{\Phi}\Phi\right) + \frac{ic}{2}, \tag{3.4.16}$$

where $c = c(t)$ is an arbitrary function of t.

Putting (3.4.14), (3.4.16) into (3.4.10), (3.4.11), we observe that the compatibility condition for (3.4.10) and (3.4.11) implies that

$$c(t) \in \mathbb{R},$$

and then (3.4.10) is equivalent to the following equation for Φ:

$$i\frac{\partial\Phi}{\partial t} = -\frac{\partial^2\Phi}{\partial x^2} + 2\kappa|\Phi|^2\Phi + c(t)\Phi. \tag{3.4.17}$$

Under the transformation

$$\Phi \mapsto \psi = \Phi \exp\left(i\int c(t)\mathrm{d}t\right),$$

this equation is reduced to NLS equation. We arrive at the following result.

Theorem 3.4.2. *If the matrix valued function* $\mathbf{U}$ *and* $\mathbf{V}$ *of the form* (3.4.2) *and* (3.4.3) *satisfy the zero curvature equation* (3.4.1)*, then*

$$\mathbf{U}_0 = \sqrt{\kappa}\begin{pmatrix} 0 & \mathrm{e}^{i\theta}\overline{\psi} \\ \mathrm{e}^{-i\theta}\psi & 0 \end{pmatrix}, \tag{3.4.18}$$

$$\mathbf{V}_0 = \begin{pmatrix} i\left(\kappa|\psi|^2 + \frac{\theta'}{2}\right) & -\sqrt{\kappa}\mathrm{e}^{i\theta}\frac{\partial\overline{\psi}}{\partial x} \\ \sqrt{\kappa}\mathrm{e}^{-i\theta}\frac{\partial\psi}{\partial x} & -i\left(\kappa|\psi|^2 + \frac{\theta'}{2}\right) \end{pmatrix}, \tag{3.4.19}$$

where $\theta = \theta(t)$ *is a real function of* t *and* $\psi = \psi(x,t)$ *is a solution of NLS equation*

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\kappa|\psi|^2\psi. \tag{3.4.20}$$

On the contrary, an arbitrary real function $\theta(t)$ *and a solution* $\psi(x,t)$ *of NLS equation define a solution* $(\mathbf{U},\mathbf{V})$ *of the zero curvature equation* (3.4.1) *by the formulae* (3.4.2)*,* (3.4.3) *and* (3.4.18)*,* (3.4.19)*.*

Therefore, this theorem says that the solutions of the zero curvature equation (3.4.1) is the class of λ-parametric matrix functions (3.4.2), (3.4.3) are parametrized by one real function and a solution of NLS equation.

Corollary 3.4.3. *The nonlinear Schrödinger equation* (3.5.1) *for* ψ *is equivalent to the zero curvature equation for the* λ*-parameter matrix functions*

$$\mathbf{U} = \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\kappa}\overline{\psi} \\ \sqrt{\kappa}\psi & \frac{-\lambda}{2i} \end{pmatrix}, \tag{3.4.21}$$

$$\mathbf{V} = \begin{pmatrix} i\kappa|\psi|^2 & -i\sqrt{\kappa}\frac{\partial\overline{\psi}}{\partial x} \\ i\sqrt{\kappa}\frac{\partial\psi}{\partial x} & -i\kappa|\psi|^2 \end{pmatrix} - \begin{pmatrix} \frac{\lambda^2}{2i} & \lambda\sqrt{\kappa}\overline{\psi} \\ \lambda\sqrt{\kappa}\psi & \frac{-\lambda^2}{2i} \end{pmatrix}. \tag{3.4.22}$$

Remark 3. *It follows from Proposition* (3.4.1) *and Corollary* (3.4.3) *that the NLS equation is equivalent to the commutativity of linear operators* $\mathscr{L}$ *and* $\mathscr{M}$ (3.4.7)*,* (3.4.8) *which define the Lax equation pair for the NLS equation [7].*

3.5 Application of the inverse problem to the NLS equation

Let us consider the dynamical system generated by the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\kappa|\psi|^2\psi, \quad (\kappa \in \mathbb{R}) \tag{3.5.1}$$

Here, $\psi(x)$ is a given complex valued function belongs to Schwartz space $\mathcal{S}(\mathbb{R})$. We shall apply the inverse problem, discussed in Chapter 3, to reconstruct the solution $\psi(x,t)$ of the Schrödinger equation from the initial data ψ.

First, we formulate the linear problem corresponding to ψ,

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x} = \mathbf{U}(x,\lambda), \qquad \mathbf{T}(x,y,\lambda)|_{x=y} = \mathbf{I}, \tag{3.5.2}$$

where

$$\begin{aligned} \mathbf{U}(x,\lambda) &= \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3 \\ &= \sqrt{\kappa}\begin{pmatrix} 0 & \overline{\psi}(x) \\ \psi(x) & 0 \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{2i} & 0 \\ 0 & -\frac{\lambda}{2i} \end{pmatrix}. \end{aligned}$$

Since $\psi(x) \in \mathcal{S}(\mathbb{R}_x)$ $\mathbf{U} \in L_1^{n\times n}$ and the system (3.5.2) is in the decreasing case which was aboard in Chapter 2. As we established in Section (**??**) the monodromy matrix $\mathbf{M}(\lambda)$ has the form

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix} \tag{3.5.3}$$

where $a(\lambda)$, $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ and $|a(\lambda)|^2 - \epsilon|a(\lambda)|^2 = 1$. From the linear problem, we derive the spectral data, proceeding in similar manner as in Section (2.2.3). As we did it in Chapter 2, we have to consider two cases: 1) if $\kappa \geq 0$ the spectral data consisting only of the function $b(\lambda)$, 2) if $\kappa \geq 0$ the spectral data is the set $\{b(\lambda), \lambda_j, \gamma_j : j = 1, 2, .., n\}$; where λ_j are the zeroes of $a(\lambda)$ on its extended analytic domain and γ_j are related with the values of the Jost solution in the λ_j. Then we define the time dependent function

$$b(t,\lambda) = \mathrm{e}^{-i\lambda^2 t} b(\lambda), \tag{3.5.4}$$

if $\kappa \geq 0$. And we add the functions

$$\gamma_j(t) = \mathrm{e}^{-i\lambda^2 t}\gamma_j \tag{3.5.5}$$

and the complex numbers λ_j if $\kappa < 0$.

For each fixed t, the functions $b(\lambda, t)$ and $\gamma_j(t)$ satisfy the conditions of the Theorem (3.1.1) or Theorem (3.1.2). Thus, there exist a function $\psi(x,t)$ such that

$$\mathbf{M}(t,\lambda) = \begin{pmatrix} a(t,\lambda) & \varepsilon\overline{b}(t,\lambda) \\ b(t,\lambda) & \overline{a}(t,\lambda) \end{pmatrix}, \tag{3.5.6}$$

where

$$a(\lambda) = \begin{cases} \prod_{i=1}^{n}\left(\frac{\lambda-\overline{\lambda}_i}{\lambda-\lambda_i}\right)\exp\left(\frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{\ln(1-|b(\lambda)|^2)}{\mu-\lambda}\mathrm{d}\mu\right) & \text{if } \varepsilon = 1 \\ \exp\left(\frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{\ln(1-|b(\lambda)|^2)}{\mu-\lambda}\mathrm{d}\mu\right) & \text{if } \varepsilon = -1, \end{cases} \tag{3.5.7}$$

is the monodromy matrix corresponding to $\psi(x,t)$, with matrix coefficient

$$\mathbf{U}(x,t,\lambda) = \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\kappa}\overline{\psi}(x,t) \\ \sqrt{\kappa}\psi(x,t) & -\frac{\lambda}{2i} \end{pmatrix}. \tag{3.5.8}$$

By the Theorem (3.2.3), if we construct the matrix

$$\mathbf{V}(x,t,\lambda) = \begin{pmatrix} i\kappa|\psi(x,t)|^2 & -i\sqrt{\kappa}\frac{\partial\overline{\psi}(x,t)}{\partial x} \\ i\sqrt{\kappa}\frac{\partial\psi(x,t)}{\partial x} & -i\kappa|\psi(x,t)|^2 \end{pmatrix} - \begin{pmatrix} \frac{\lambda^2}{2i} & \lambda\sqrt{\kappa}\overline{\psi}(x,t) \\ \lambda\sqrt{\kappa}\psi(x,t) & \frac{-\lambda^2}{2i} \end{pmatrix}. \tag{3.5.9}$$

$\mathbf{U}$ and V satisfy the zero curvature equation (3.4.1) for all λ. Now, It follows from corollary (3.4.3) that $\psi(x,t)$ is a solution of the NLS equation with $\psi(x,0) = \psi(x)$.

Summarizing, for reconstruct the solution $\psi(x,t)$ of NLS equation that satisfy the initial condition $\psi(x)$, we begin with the linear problem corresponding to ψ and derive the spectral data, depending of sign of κ. Using these data, we define the time dependent functions (3.5.4) and (3.5.5). For each t we apply the inverse problem analyzed in chapter 3, and we get the complex valued function $\psi(x,t)$ that is the solution on the NLS equation with the initial condition $\psi(x)$.

We present the method for solving the initial value problem for the NLS equation in the following commutative diagram

$$
\begin{array}{ccc}
\psi(x) & \xrightarrow{\text{Linear problem}} & \left\{ \begin{array}{cc} b(\lambda) & if\ \kappa \geq 0 \\ \{b(\lambda), \lambda_j, \gamma_j : j = 1,2,..,n\} & if\ \kappa \geq 0 \end{array} \right. \\
\downarrow & & \downarrow \\
\psi(x,t) & \xleftarrow{\text{Inverse problem}} & \left\{ \begin{array}{cc} b(t,\lambda) & if\ \kappa \geq 0 \\ \{b(t,\lambda), \lambda_j, \gamma_j(t) : j = 1,2,..,n\} & if\ \kappa \geq 0 \end{array} \right.
\end{array}
$$

For to complete our analysis, we give some observation about the case $\kappa = 0$. Firstly, the NLS equation reduce to the linear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2}. \tag{3.5.10}$$

Let us consider the asymptotic behavior of the transition coefficient $a(\lambda)$ and $b(\lambda)$ as $\kappa \to 0$. We observe that the integral representation (2.2.44) for the Lost solution $\mathbf{T}_-(x,\lambda)$

$$\mathbf{T}_-(x,\lambda) = \mathbf{E}(x,\lambda) + \int_{-\infty}^{x} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z,\lambda)\mathrm{d}z + O(|\kappa|). \tag{3.5.11}$$

If the variable x tends to infinity in this formula we get

$$a(\lambda) = 1 + O(|\kappa|), \qquad b(\lambda) = \sqrt{\kappa}\int_{-\infty}^{\infty} \psi(x)\mathrm{e}^{-i\lambda x}\mathrm{d}x + O(|\kappa|). \tag{3.5.12}$$

As we can see the discrete spectrum disappears and the linear problem (3.5.2) can be interpreted as the Fourier transform. Moreover, the time dynamics of $b(\lambda)$ is given by the Fourier transform of $\psi(x,y)$ subjet to ψ. Therefore, in the general case $\kappa \neq 0$, the inverse problem is interpreted as a nonlinear analogue of the Fourier method.

3.6 Non solitonic and solitonic solutions of NLS equation

In this part we shall derive formulae for soliton solution of the NLS equation. Follow [7], the soliton is a wave solution with the follow condition

1. Propagation does not change its shape.
2. It has finite energy and all the integral of motion are finite.

In the physics literature the term soliton sometimes refers to a particle solution.

One important mean for the transition coefficient is that the ratio $\frac{b(\lambda)}{a(\lambda)}$ is the reflection coefficient. If

$$b \equiv 0, \tag{3.6.1}$$

the linear problem is referred as reflectionless. Now, we assume (3.6.1) and find soliton solution for NLS equation. Let us only only consider $\kappa < 0$ because $\kappa > 0$ produces trivial solutions.

We begin with a single zero λ_1, $\mathrm{Im}\lambda_1$ and one complesx coefficient $\gamma_1 \neq 0$. From Chapter 3, under these conditions, que have that

$$\psi(x) = \frac{2\mathrm{Im}\lambda_1}{\sqrt{\kappa}} \frac{\gamma_1(x)}{1 + |\gamma_1(x)|^2}, \quad \text{with } \gamma_1(x) = \mathrm{e}^{-i\lambda x}\gamma_1. \tag{3.6.2}$$

Applying the inverse problem, we get the solution of NLS equation

$$\psi(x,t) = \frac{2\mathrm{Im}\lambda_1}{\sqrt{\kappa}} \frac{\gamma_1(x,t)}{1 + |\gamma_1(x,t)|^2}, \quad \text{where } \gamma_1(x) = \mathrm{e}^{-i\lambda_1^2 x}\gamma_1. \tag{3.6.3}$$

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